Jacobiova metoda

Princip:

Z i-té rovnice vyjádříme i-tou složku vektoru $\mathbf{x}$

i-tá rovnice: $$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i$$

pro $a_{ii} \neq 0$: $$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1\ j\neq i}^{n} a_{ij}x_j\right)$$

Iterační formule:

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1\ j\neq i}^{n} a_{ij}x_j^{(k)}\right)}$$

Př 1.

Jacobiova metoda pro reseni soustavy Ax=b, kde

```
A =
     6     3    -1
     1     5    -2
     1    -3     4

b =
     5
     3
     1

x0 =
     0
     0
     0

H =
         0   -0.5000    0.1667
   -0.2000         0    0.4000
   -0.2500    0.7500         0

g =
    0.8333
    0.6000
    0.2500

vl.cisla H =

    0.6308
   -0.0708
   -0.5601

1. iterace = [ 0.833333], rozdil iteraci [ 0.833333]
             [ 0.600000],                [0.600000]
             [ 0.250000],                [0.250000]

2. iterace = [ 0.575000], rozdil iteraci [ -0.258333]
             [ 0.533333],                [-0.066667]
             [ 0.491667],                [0.241667]

3. iterace = [ 0.648611], rozdil iteraci [ 0.073611]
             [ 0.681667],                [0.148333]
             [ 0.506250],                [0.014583]

4. iterace = [ 0.576875], rozdil iteraci [ -0.071736]
             [ 0.672778],                [-0.008889]
             [ 0.599097],                [0.092847]

5. iterace = [ 0.596794], rozdil iteraci [ 0.019919]
             [ 0.724264],                [0.051486]
             [ 0.610365],                [0.011267]


x =

    0.5968
    0.7243
    0.6104
```

$\varepsilon = 0{,}1$

Gaussova-Seidelova metoda

Princip:

Stejný jako u Jacobiovy metody s tím rozdílem, že jestliže při výpočtu $(k+1)$-iterace již známe $(k+1)$-iteraci některých složek, tak ji použijeme.

Iterační formule:

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)}$$

Př. 2

Gauss-Seidelova metoda pro reseni soustavy Ax=b, kde

```
A =
      6     3    -1
      1     5    -2
      1    -3     4

b =
      5
      3
      1

x0 =
      0
      0
      0

H =
           0   -0.5000    0.1667
           0    0.1000    0.3667
           0    0.2000    0.2333

g =
    0.8333
    0.4333
    0.3667

vl.cisla H =

         0
   -0.1122
    0.4456

1. iterace = [ 0.833333], rozdil iteraci [ 0.833333]
             [ 0.433333],                [0.433333]
             [ 0.366667],                [0.366667]

2. iterace = [ 0.677778], rozdil iteraci [ -0.155556]
             [ 0.611111],                [0.177778]
             [ 0.538889],                [0.172222]

3. iterace = [ 0.617593], rozdil iteraci [ -0.060185]
             [ 0.692037],                [0.080926]
             [ 0.614630],                [0.075741]

4. iterace = [ 0.589753], rozdil iteraci [ -0.027840]
             [ 0.727901],                [0.035864]
             [ 0.648488],                [0.033858]


x =

    0.5898
    0.7279
    0.6485
```

$\varepsilon = 0{,}1$

Relaxační metoda SOR

Princip:

Vyjdeme z Gaussovy-Seidelovy metody jejíž iterační formuli lze psát takto:

$$x_i^{(k+1)} = x_i^{(k)} + r_i^{(k)} \quad \text{kde}$$

$$r_i^{(k)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij} x_j^{(k)} \right)$$

Abychom urychlili výpočet, nebudeme přičítat $r_i^{(k)}$, ale $\omega r_i^{(k)}$, tj.

$$x_i^{(k+1)} = x_i^{(k)} + \omega r_i^{(k)}$$

Iterační formule:

$$\boxed{x_i^{(k+1)} = x_i^{(k)} + \omega \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij} x_j^{(k)} \right)}$$

Poznámka:
$(k+1)$-iterace metody SOR je lineární kombinací $(k+1)$-iterace získané Gauss-Seidlovou metodou a předchozí k-té iterace.

$$x_i^{(k+1)} = \omega \underbrace{\frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right)}_{x_i^{(k+1)} \text{ z Gaussovy-Seidelovy metody}} + (1-\omega) x_i^{(k)}$$

Př 3

Relaxacni metoda SOR pro reseni soustavy Ax=b, kde

```
A =
     6     3    -1
     1     5    -2
     1    -3     4

b =
     5
     3
     1

x0 =
     0
     0
     0

omega =
    1.2000

H =
   -0.2000   -0.6000    0.2000
    0.0480   -0.0560    0.4320
    0.1032    0.1296    0.1288

g =
    1.0000
    0.4800
    0.4320

vl.cisla H =

   0.1224 + 0.0807i
   0.1224 - 0.0807i
  -0.3721

1. iterace = [ 1.000000], rozdil iteraci [ 1.000000]
             [ 0.480000],                [0.480000]
             [ 0.432000],                [0.432000]

2. iterace = [ 0.598400], rozdil iteraci [ -0.401600]
             [ 0.687744],                [0.207744]
             [ 0.653050],                [0.221050]

3. iterace = [ 0.598284], rozdil iteraci [ -0.000116]
             [ 0.752327],                [0.064583]
             [ 0.666999],                [0.013950]


x =

    0.5983
    0.7523
    0.6670
```

$\varepsilon = 0{,}1$

```
opt_omega =

    1.1262
```

Př 4a

Jacobiova metoda pro reseni soustavy Ax=b, kde

konvergují

```
A =
     1     2    -2
     1     1     1
     2     2     1

b =
     1
     3
     5

x0 =
     0
     0
     0

H =
     0    -2     2
    -1     0    -1
    -2    -2     0

g =
     1
     3
     5

vl.cisla H =

   1.0e-05 *

  -0.4588 + 0.7946i
  -0.4588 - 0.7946i
   0.9175

1. iterace = [ 1.000000], rozdil iteraci [ 1.000000]
             [ 3.000000],                [3.000000]
             [ 5.000000],                [5.000000]

2. iterace = [ 5.000000], rozdil iteraci [ 4.000000]
             [ -3.000000],                [-6.000000]
             [ -3.000000],                [-8.000000]

3. iterace = [ 1.000000], rozdil iteraci [ -4.000000]
             [ 1.000000],                [4.000000]
             [ 1.000000],                [4.000000]

4. iterace = [ 1.000000], rozdil iteraci [ 0.000000]
             [ 1.000000],                [0.000000]
             [ 1.000000],                [0.000000]

x =
     1
     1
     1
```

Př 4b

Gauss-Seidelova metoda pro reseni soustavy Ax=b, kde

diverguje

```
A =
      1     2    -2
      1     1     1
      2     2     1

b =
      1
      3
      5

x0 =
      0
      0
      0

H =
      0    -2     2
      0     2    -3
      0     0     2

g =
      1
      2
     -1

vl.cisla H =
      0
      2
      2

1. iterace = [ 1.000000], rozdil iteraci [ 1.000000]
             [ 2.000000],                [2.000000]
             [ -1.000000],                [-1.000000]

2. iterace = [ -5.000000], rozdil iteraci [ -6.000000]
             [ 9.000000],                [7.000000]
             [ -3.000000],                [-2.000000]

3. iterace = [ -23.000000], rozdil iteraci [ -18.000000]
             [ 29.000000],                [20.000000]
             [ -7.000000],                [-4.000000]

4. iterace = [ -71.000000], rozdil iteraci [ -48.000000]
             [ 81.000000],                [52.000000]
             [ -15.000000],                [-8.000000]

5. iterace = [ -191.000000], rozdil iteraci [ -120.000000]
             [ 209.000000],                [128.000000]
             [ -31.000000],                [-16.000000]

6. iterace = [ -479.000000], rozdil iteraci [ -288.000000]
             [ 513.000000],                [304.000000]
             [ -63.000000],                [-32.000000]
```

Pi5a

Jacobiova metoda pro reseni soustavy Ax=b, kde

divergují

```
A =
     5     3     4
     2     5     4
     1     4     5

b =
    12
    11
    10

x0 =
     0
     0
     0

H =
         0   -0.6000   -0.8000
   -0.4000         0   -0.8000
   -0.2000   -0.8000         0

g =
    2.4000
    2.2000
    2.0000

vl.cisla H =

   -1.1592
    0.4000
    0.7592

1. iterace = [ 2.400000], rozdil iteraci [ 2.400000]
             [ 2.200000],                [2.200000]
             [ 2.000000],                [2.000000]

2. iterace = [ -0.520000], rozdil iteraci [ -2.920000]
             [ -0.360000],                [-2.560000]
             [ -0.240000],                [-2.240000]

3. iterace = [ 2.808000], rozdil iteraci [ 3.328000]
             [ 2.600000],                [2.960000]
             [ 2.392000],                [2.632000]

4. iterace = [ -1.073600], rozdil iteraci [ -3.881600]
             [ -0.836800],                [-3.436800]
             [ -0.641600],                [-3.033600]

5. iterace = [ 3.415360], rozdil iteraci [ 4.488960]
             [ 3.142720],                [3.979520]
             [ 2.884160],                [3.525760]

6. iterace = [ -1.792960], rozdil iteraci [ -5.208320]
             [ -1.473472],                [-4.616192]
             [ -1.197248],                [-4.081408]

7. iterace = [ 4.241882], rozdil iteraci [ 6.034842]
             [ 3.874982],                [5.348454]
             [ 3.537370],                [4.734618]
```

Př 5b

Gauss-Seidelova metoda pro reseni soustavy Ax=b, kde

konverguje

```
A =
      5     3     4
      2     5     4
      1     4     5

b =
     12
     11
     10

x0 =
      0
      0
      0

H =
              0   -0.6000   -0.8000
              0    0.2400   -0.4800
              0   -0.0720    0.5440

g =
    2.4000
    1.2400
    0.5280

vl.cisla H =

         0
    0.1519
    0.6321

1. iterace = [ 2.400000], rozdil iteraci [ 2.400000]
             [ 1.240000],                [1.240000]
             [ 0.528000],                [0.528000]

2. iterace = [ 1.233600], rozdil iteraci [ -1.166400]
             [ 1.284160],                [0.044160]
             [ 0.725952],                [0.197952]

3. iterace = [ 1.048742], rozdil iteraci [ -0.184858]
             [ 1.199741],                [-0.084419]
             [ 0.830458],                [0.104506]

4. iterace = [ 1.015788], rozdil iteraci [ -0.032954]
             [ 1.129318],                [-0.070424]
             [ 0.893388],                [0.062930]

5. iterace = [ 1.007699], rozdil iteraci [ -0.008090]
             [ 1.082210],                [-0.047108]
             [ 0.932692],                [0.039304]

6. iterace = [ 1.004520], rozdil iteraci [ -0.003179]
             [ 1.052038],                [-0.030172]
             [ 0.957465],                [0.024773]

7. iterace = [ 1.002805], rozdil iteraci [ -0.001715]
             [ 1.032906],                [-0.019132]
             [ 0.973114],                [0.015649]

8. iterace = [ 1.001765], rozdil iteraci [ -0.001040]
             [ 1.020802],                [-0.012103]
             [ 0.983005],                [0.009891]

9. iterace = [ 1.001114], rozdil iteraci [ -0.000650]
             [ 1.013150],                [-0.007652]
```

Př 6

```
Jacobiova metoda pro reseni soustavy  Ax=b, kde

A =
     8     4     2
     1    10     1
     0     0     2

b =
    14
    12
     2

x0 =
     0
     0
     0

H =
         0   -0.5000   -0.2500
   -0.1000         0   -0.1000
         0         0         0

g =

    1.7500
    1.2000
    1.0000

vl.cisla H =

    0.2236
   -0.2236
         0

1. iterace = [ 1.750000], rozdil iteraci [ 1.750000]
             [ 1.200000],                [1.200000]
             [ 1.000000],                [1.000000]

2. iterace = [ 0.900000], rozdil iteraci [ -0.850000]
             [ 0.925000],                [-0.275000]
             [ 1.000000],                [0.000000]

3. iterace = [ 1.037500], rozdil iteraci [ 0.137500]
             [ 1.010000],                [0.085000]
             [ 1.000000],                [0.000000]

4. iterace = [ 0.995000], rozdil iteraci [ -0.042500]
             [ 0.996250],                [-0.013750]
             [ 1.000000],                [0.000000]

5. iterace = [ 1.001875], rozdil iteraci [ 0.006875]
             [ 1.000500],                [0.004250]
             [ 1.000000],                [0.000000]

x =
    1.0019
    1.0005
    1.0000
```

ODHAD CHYBY

IF $\boxed{\| H \| \leq q < 1}$:

$$\boxed{\| x^{(k)} - x^* \| \leq \frac{q}{1-q} \underbrace{\| x^{(k)} - x^{(k-1)} \|}_{\leq \varepsilon}}$$

normy:

vektorová	maticová
$\|x\|_1 = \sum_i \|x_i\|$	$\|A\|_S$
$\|x\|_\infty = \max_i \|x_i\|$	$\|A\|_R$
$\|x\|_2 = (\sum_i x_i^2)^{\frac{1}{2}}$	$\|A\|_{SP}^2 = \max \lambda(A^H A)$

Př 6

$$H = \begin{bmatrix} 0 & -0{,}5 & -0{,}25 \\ -0{,}1 & 0 & -0{,}1 \\ 0 & 0 & 0 \end{bmatrix} \qquad x_5 - x_4 = \begin{bmatrix} 0{,}006875 \\ 0{,}004250 \\ 0{,}000000 \end{bmatrix}$$

---

a) $\|H\|_S = 0{,}5 \quad \longleftrightarrow \quad \| x_5 - x_4 \|_1 = 0{,}011125$

b) $\|H\|_R = 0{,}75 \quad \longleftrightarrow \quad \| x_5 - x_4 \|_\infty = 0{,}006875$

c) $\|H\|_{SP} \doteq 0{,}56 \quad \longleftrightarrow \quad \| x_5 - x_4 \|_2 \doteq 0{,}0081$

$\uparrow$ q $\qquad\qquad$ $\uparrow$ ε

$$\| x_5 - x^* \| \leq \frac{q}{1-q} \varepsilon$$

a) $\| \cdot \|_1 \qquad 0{,}011125$

b) $\| \cdot \|_\infty \qquad \frac{0{,}75}{0{,}25} \cdot 0{,}006875 = 0{,}0206$

c) $\| \cdot \|_2 \qquad \frac{0{,}56}{0{,}44} \cdot 0{,}0081 = 0{,}0103$