8.34 × 8.72

D H

$\boxed{\text{GIRG}}$ II. PP.

$$\lim_{x\to 0} \frac{\overbrace{\ln(\sin^2 x+1)}^{\boxed{I}} + \overbrace{2x^3 \operatorname{arctg}\left(-\frac{1}{x}\right)}^{\boxed{II}}}{\underbrace{x\cdot \operatorname{tg} x + 3x^2}_{\boxed{III}}} = ?$$

- existence limity
- hodnota limity

X – Cauchyova def. lim. $\forall \varepsilon > 0 \ldots\ldots \Rightarrow |f(x)-b| < \varepsilon$

X – topol. verze

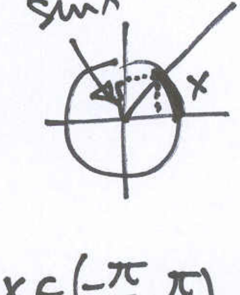
X – Heineho def. limity

$\boxed{I}$

$$\lim_{x\to 0} \ln(\sin^2 x+1) = \lim_{x\to 0} g(f(x))$$

$g(y) = \ln y$

$f(x) = \sin^2 x + 1$ $\boxed{\lim_{x\to 0} f(x) = 1}$



$\lim_{x\to 0} \sin x = 0$, $\underline{0 \le |\sin x| \le \overset{x\to 0}{\longrightarrow 0}{|x|}}$ pro $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Downarrow$ ALF

(+) věta o srovnání $\Rightarrow \lim_{x\to 0} |\sin x| = 0$

$\sin^2 x + 1 \xrightarrow[x\to 0]{} 1$

$\lim_{x\to 0} \sin x = 0 \Leftarrow$

$$\lim_{y\to 1} g(y) = \lim_{y\to 1} \ln y = 0 \qquad \lim_{x\to 0} g(\underbrace{\vec{f}(x)}_{\to 1})$$

Cauch. def. limity

$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D(g): 0 < |x - x_0| < \delta \Rightarrow$

$\Rightarrow |\ln x - 0| < \varepsilon$

$\delta = \delta(\varepsilon)$

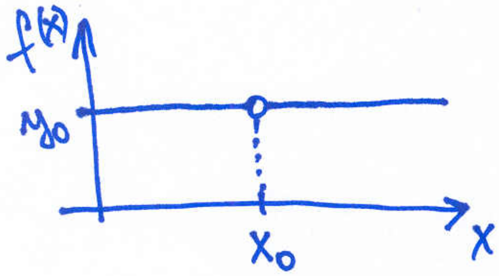
$$\lim_{x\to 0} f(x) = 1 \qquad \lim_{y\to 1} g(y) = 0$$

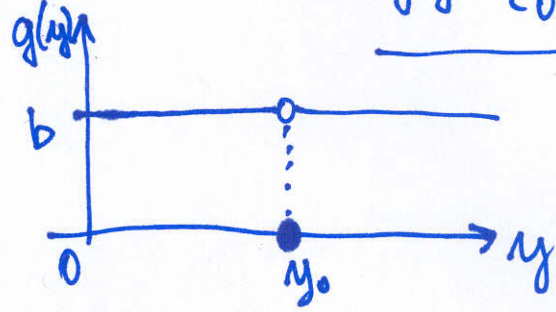
$$?\Downarrow?$$

$$\lim_{x\to 0} g(f(x)) \overset{?}{=} \lim_{y\to \lim\limits_{x\to 0} f(x)} g(y) = 0$$

• protipříklad

$$g(y) = \begin{cases} b & \text{pro } y \neq y_0 \\ 0 & \text{pro } y = y_0 \end{cases}$$





$$\lim_{x\to x_0} f(x) = y_0 \qquad \lim_{y\to y_0} g(y) = b$$

$$\not\parallel$$

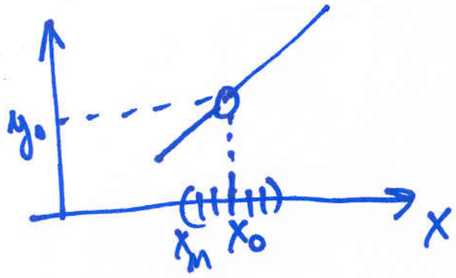
$x \neq x_0 : f(x) = y_0 \qquad \lim_{x\to x_0} g(f(x)) = \lim_{x\to x_0} g(y_0) = 0$

$x \neq x_0: \ g(f(x)) = g(y_0) = 0$

- dodatečné předpoklady na f, g, aby platila rovnost

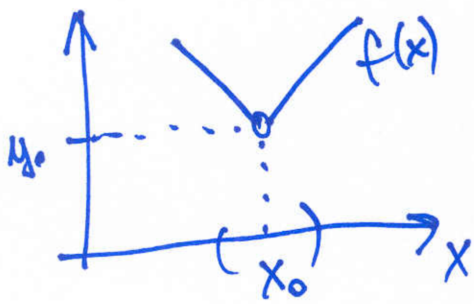
f – před.

f – prostá



$$\lim_{x\to x_0} g(f(x)) = b$$

$$f(x_n) \neq y_0$$



$$\boxed{\exists P(x_0)\ \forall x \in P(x_0) \cap D(f) : f(x) \neq y_0}$$

$$\lim_{x\to x_0} g(f(x)) = b = \lim_{y\to y_0} g(y)$$

g – předef.

$$\boxed{g(y_0) := b}$$

$$\lim_{x\to x_0} g(f(x)) = b$$

$$g(y_0) = b$$

<u>Věta (limita složené funkce)</u>: Nechť $f : D \subset \mathbb{R} \to \mathbb{R}$, $f : D \xrightarrow{na} H$, $g : \mathcal{H} \to \mathcal{M}$, $H \subset \mathcal{H}$.

Nechť $\exists \lim_{x\to x_0} f(x) =: y_0$, $\exists \lim_{y\to y_0} g(y) =: b$.

$$\left(\exists P(x_0)\ \forall x \in P(x_0) \cap D : f(x) \neq y_0\right) \vee \left(g(y_0) = b\right) \implies \exists \lim_{x\to x_0} g(f(x)) = b.$$

add $\boxed{\text{I}}$

$$\lim_{x\to 0} g(f(x)) \overset{?}{=} \lim_{y\to 1} g(y) = \lim_{y\to 1} \ln y$$

$$g(y_0) = g(1) = \ln 1 = \underline{\underline{0 = b}}$$ – splněn předp. týk. se fce g

VLSF $\Rightarrow$ $\boxed{\lim_{x\to 0} \ln(\sin^2 x + 1) = 0}$

$\boxed{\text{II}}$ $\lim_{x\to 0} 2x^3 \operatorname{arctg}\left(-\frac{1}{x}\right) = 0$

$$0 \le \left|2x^3 \operatorname{arctg}\left(-\frac{1}{x}\right)\right| =$$

$$= 2|x|^3 \cdot \left|\operatorname{arctg}\left(-\frac{1}{x}\right)\right| \le \underbrace{2|x|^3 \cdot \frac{\pi}{2}}_{} \xrightarrow[x\to 0]{} 0$$

+ srovnávací věta — dle ALF

přestože $\nexists \lim_{x\to 0} \operatorname{arctg}\left(-\frac{1}{x}\right)$

proč? $f(x) = -\frac{1}{x}$, $g(y) = \operatorname{arctg} y$

$\nexists \lim_{x\to 0} f(x)$, $x_n = \frac{1}{n} \to 0$, $f(x_n) = n \xrightarrow{n\to+\infty} +\infty$

$y_n = -\frac{1}{n} \to 0$, $f(y_n) = -n \xrightarrow{n\to+\infty} -\infty$

$$\left.\begin{array}{l} \{x_n\} \text{ lib. posl.},\ x_n > 0,\ \frac{1}{x_n} \to +\infty,\ x_n \to 0 \\ \{y_n\} \text{ lib. posl.},\ y_n < 0,\ \frac{1}{y_n} \to -\infty,\ y_n \to 0 \end{array}\right\} \Rightarrow \nexists \lim_{x\to 0} f(x)$$

Definice (jednostranné limity dle "Heineho"):

Nechť $f: D \subset \mathbb{R} \to \mathbb{R}$, x_0 je hromadný bod $D(f)$. $b \in \mathbb{R}$

Řekneme, že fce f má $\begin{cases} \text{pravostrannou limitu} \\ \text{levostrannou limitu} \end{cases}$ b v bodě x_0, pokud

$$\forall \{x_n\} \subset D(f): \left(\forall n \in \mathbb{N}: x_n \neq x_0, \begin{cases} x_n > x_0 \\ x_n < x_0 \end{cases}, x_n \xrightarrow{n \to +\infty} x_0\right) \Rightarrow f(x_n) \to b.$$

Zapisujeme: $\begin{cases} \lim\limits_{x \to x_0+} f(x) = b \quad \text{či} \quad f(x_0+) = b \\ \lim\limits_{x \to x_0-} f(x) = b \quad \text{či} \quad f(x_0-) = b \end{cases}$

Poznámky: 1) jednostranné limity lze def. i pro $b = \pm\infty$, zároveň i Cauchyovou či topologickou verzí.

2) platí "jednostranné" verze tvrzení:
- jednoznačnost limity
- algebra limit
- srovnávací věty
- omezenost a limita

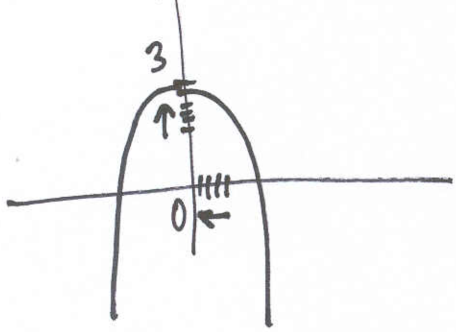
3) ve větě o LSF lze uvažovat $x \to x_0$ i $\begin{matrix} x \to x_0+ \\ x \to x_0- \end{matrix}$,

nelze však obecně omezit předp. $\exists \lim\limits_{y \to y_0} g(y)$

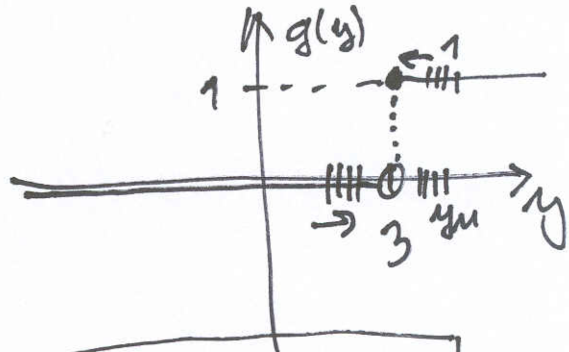
na $\exists \lim\limits_{\substack{y \to y_0+ \\ y \to y_0-}} g(y)$.

ilustrační příklad:

$f(x) = 3 - x^2$



$$g(y) = \begin{cases} 1 & \text{pro } y \geq 3 \\ 0 & \text{pro } y < 3 \end{cases}$$



$$\lim_{x \to 0+} f(x) = 3$$

$$\boxed{\lim_{y \to 3+} g(y) = 1}$$

$$?\Downarrow?$$

$$\lim_{x \to 0+} g(f(x)) \overset{?}{=} 1 \quad \text{NE}$$

$\{x_n\}$ je lib. p.

$$x_n \to 0+ \Rightarrow f(x_n) = 3 - x_n^2 \to 3- \Rightarrow$$

$$3 - x_n^2 < 3$$

$$\Rightarrow g(f(x_n)) = 0 \Rightarrow \lim_{n \to +\infty} g(f(x_n)) = \underline{\underline{0}}$$

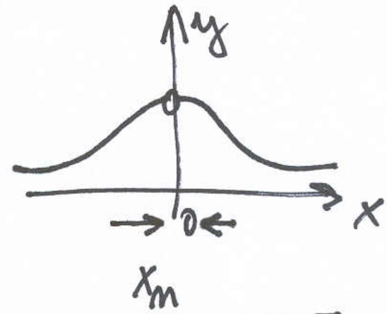
$$\nexists \lim_{y \to 3} g(y)$$

Věta (kritérium existence oboustranné limity):

Nechť $f: D \subset \mathbb{R} \to \mathbb{R}$, x_0 je hromadný bod $D(f)$.

$$\exists \lim_{x \to x_0} f(x) = b \iff \exists f(x_0+), \exists f(x_0-), f(x_0+) = f(x_0-) = b.$$

Důkaz: zřejmý



$\boxed{\Rightarrow}$ přímý

$\{x_n\} \subset D(f)$, $x_n \neq x_0$, $x_n > x_0$, $x_n \to x_0$ $\xRightarrow{\boxed{\lim_{x \to x_0} f(x) = b}}$ $f(x_n) \to b$

$x_n < x_0$ $\Rightarrow$ $f(x_n) \to b$

$\boxed{\Leftarrow}$ přímý

$\{x_n\} \subset D(f)$, $\underline{x_n \neq x_0}$, $x_n \to x_0$, $\underline{\exists f(x_0+), \exists f(x_0-)}$

$f(x_0-) = f(x_0+) = b$

$? \Downarrow ?$

$f(x_n) \to b$

možnosti:
- $x_n > x_0$ pro s.v.n. $\xRightarrow{\exists f(x_0+) = b}$ $f(x_n) \to b$
- $x_n < x_0$ pro s.v.n. $\xRightarrow{\exists f(x_0-) = b}$ $f(x_n) \to b$
- $\underline{\{x_n\} = \{x_{n_k}\} \cup \{x_{n_l}\},}$ $\left.\begin{array}{l} x_{n_k} > x_0 \xRightarrow{\exists f(x_0+) = b} f(x_{n_k}) \to b \\ x_{n_l} < x_0 \xRightarrow{\exists f(x_0-) = b} f(x_{n_l}) \to b \end{array}\right\}$

$\boxed{\Rightarrow}$ $f(x_n) \xrightarrow{n \to +\infty} b$

---

add $\boxed{\text{II}}$ $\lim\limits_{x\to 0} 2x^3 \operatorname{arctg}\left(-\frac{1}{x}\right) = 0$

$$\nexists \lim_{x\to 0} -\frac{1}{x}$$

$$\nexists \lim_{x\to 0} \operatorname{arctg}\left(-\frac{1}{x}\right) \quad \Uparrow$$

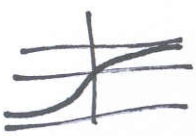
$$\lim_{x\to 0+} \operatorname{arctg}\left(-\frac{1}{x}\right) = -\frac{\pi}{2} \neq \lim_{x\to 0-} \operatorname{arctg}\left(-\frac{1}{x}\right) = \frac{\pi}{2}$$

$x_n \to 0+ \Rightarrow -\frac{1}{x_n} \to -\infty$

$\Rightarrow \operatorname{arctg}\left(-\frac{1}{x_n}\right) \to -\frac{\pi}{2}$

$x_n \to 0- \Rightarrow -\frac{1}{x_n} \to +\infty$

$\Rightarrow \operatorname{arctg}\left(-\frac{1}{x_n}\right) \to \frac{\pi}{2}$



? $\boxed{\lim\limits_{y\to +\infty} \operatorname{arctg} y = \frac{\pi}{2}}$? $\quad g(y)=\operatorname{arctg} y$

Cauchyova def. limity

$$\boxed{\forall \varepsilon>0}\ \underline{\exists \delta>0}\ \forall y\in D(g): \boxed{\delta<y} \Rightarrow \left|g(y)-\frac{\pi}{2}\right|<\varepsilon$$

$$\left|\operatorname{arctg} y-\frac{\pi}{2}\right|<\varepsilon \Rightarrow \frac{\pi}{2}-\operatorname{arctg} y<\varepsilon$$

$$\frac{\pi}{2}-\varepsilon<\operatorname{arctg} y \qquad (\operatorname{tg}(.))$$

$$0<\varepsilon<\frac{\pi}{2} \qquad \Downarrow$$

$$\operatorname{tg}\left(\frac{\pi}{2}-\varepsilon\right)<\operatorname{tg}\operatorname{arctg} y$$

$$\boxed{\underbrace{\operatorname{tg}\left(\frac{\pi}{2}-\varepsilon\right)}<y}$$

$$\boxed{\delta:=\operatorname{tg}\left(\frac{\pi}{2}-\varepsilon\right)}$$

$\boxed{\text{III}}$

$$\lim_{x\to 0}\left(\underbrace{x\,\mathrm{tg}\,x}+\underbrace{3x^2}_{\to 0}\right)=0$$

$$0\le |x\,\mathrm{tg}\,x|\le |x|\,|\mathrm{tg}\,x| = |x|\cdot\frac{|\sin x|}{|\cos x|}\le \underbrace{\frac{2}{\sqrt{2}}}_{K}\cdot|x| \to 0$$

pro $x\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$

$$\Leftarrow \quad \lim_{x\to 0} x\,\mathrm{tg}\,x = 0$$

$$\lim_{x\to 0}\frac{\ln(\sin^2 x+1)+2x^3\,\mathrm{arctg}\left(-\frac{1}{x}\right)}{x\,\mathrm{tg}\,x+3x^2}$$

neurčitý výraz "$\frac{0}{0}$" – nelze užít ALF !

Poznámka: Pokud $\exists \lim_{x\to x_0}\frac{f(x)}{g(x)}=1$, potom řekneme, že f je asymptoticky rovna fci g v bodě x_0.

Zapisujeme $\boxed{f\sim g \text{ pro } x\to x_0.}$

$\left(f(x)\equiv g(x),\ \frac{f(x)}{g(x)}=1\to 1\right)\ \left(f(x)<g(x),\ x\neq x_0\right)$

Cvičení: $\boxed{x\to 0}$

$$\boxed{\begin{gathered}x\sim\sin x\sim\mathrm{tg}\,x\sim\sinh x\sim\tanh x\sim\\ \sim\arcsin x\sim\mathrm{arctg}\,x\sim\mathrm{argsinh}\,x\sim\mathrm{argtanh}\,x\sim\\ \sim\ln(1+x)\sim e^x-1\end{gathered}}$$

$x \sim \ln(1+x)$, $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$

$x \sim \sin x$, $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

$x \sim tg x$, $\lim\limits_{x\to 0} \frac{tg x}{x} = 1$

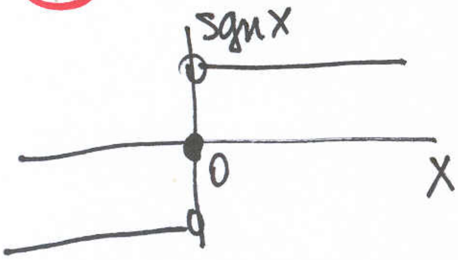
$$\lim_{x\to 0} \frac{\ln(\sin^2 x+1) + 2x^3 \operatorname{arctg}\left(-\frac{1}{x}\right)}{x\, tg x + 3x^2} = \lim_{x\to 0} \frac{\frac{\ln(\sin^2 x+1)}{\sin^2 x}\cdot \sin^2 x + 2x^3 \operatorname{arctg}\left(-\frac{1}{x}\right)}{\frac{tg x}{x}\cdot x^2 + 3x^2}$$

$$= \lim_{x\to 0} \frac{\overbrace{\frac{\ln(\sin^2 x+1)}{\sin^2 x}}^{\to 1}\cdot \left(\overbrace{\frac{\sin x}{x}}^{\to 1}\right)^2 + \overbrace{2x \operatorname{arctg}\left(-\frac{1}{x}\right)}^{\to 0}}{\underbrace{\frac{tg x}{x}}_{\to 1} + 3\cancel{x}} = \boxed{\frac{1}{4}}$$

$\sin^2 x \to 0$

Definice: 1) Částečnou limitou $c \in \mathbb{R}$ fce f pro $x \to x_0$, rozumíme takové číslo, pro které platí

$\exists \{x_n\} \subset D(f)$ taková, že $\forall n \in \mathbb{N}: x_n \neq x_0,\ x_n \to x_0,\ f(x_n) \to c$.

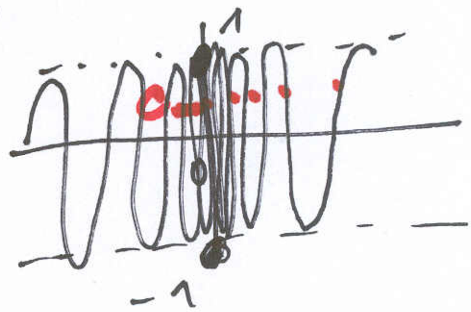


$\lim\limits_{x\to 0+} \operatorname{sgn} x = 1$ – částečná limita

$x_n = \frac{1}{n}$

$\lim\limits_{x\to 0-} \operatorname{sgn} x = -1$ – částečná limita

$x_n = -\frac{1}{n}$



$\nexists \lim\limits_{x\to 0} \cos\frac{1}{x}$

$\nexists \lim\limits_{x\to 0+} \cos\frac{1}{x}$

libovolná hodnota $\langle -1, 1\rangle \ni c$ je částečnou limitou

$$\boxed{\{x_n\} = \frac{1}{\arccos c + 2n\pi}} \longrightarrow 0+$$

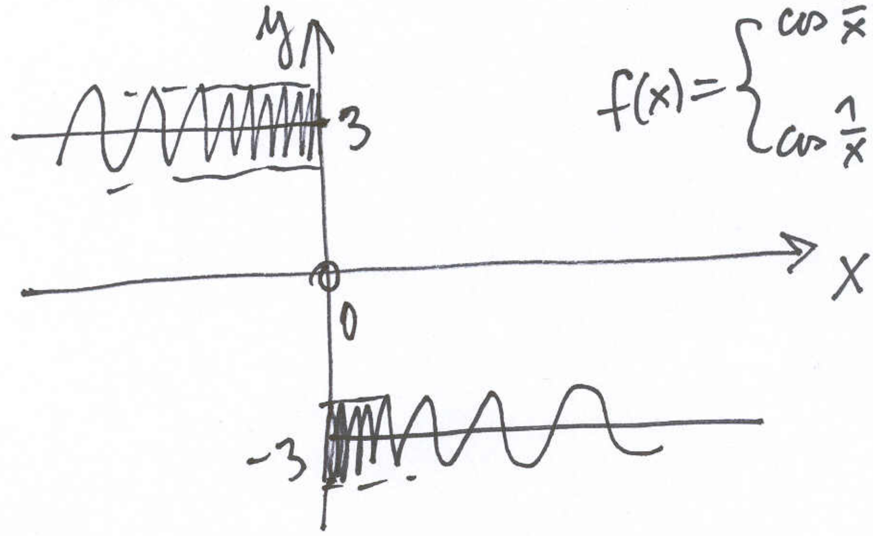
$$\cos\frac{1}{x_n} = \cos(\arccos c + 2n\pi) =$$

$$= \cos\arccos c = c \xrightarrow{n\to+\infty} c$$

2) Největší (nejmenší) částečná limita fce f ~~pro $x \to x_0$~~ se nazývá horní (dolní) limita fce f a značíme

$$\limsup_{x \to x_0} f(x) = \overline{\lim_{x \to x_0}}\, f(x)$$

$$\liminf_{x \to x_0} f(x) = \underline{\lim}_{x \to x_0}\, f(x)$$



$$f(x) = \begin{cases} \cos \frac{\wedge}{\hat{x}} + 3 & x < 0 \\ \cos \frac{1}{x} - 3 & x > 0 \end{cases}$$

$$\nexists \lim_{x \to 0} f(x), \quad \nexists \lim_{\substack{x \to 0+ \\ x \to 0-}} f(x)$$

$$\boxed{\begin{aligned} \limsup_{x \to 0} f(x) &= 4 \\ \liminf_{x \to 0} f(x) &= -4 \end{aligned}}$$