1 Matematická analýza II.

1.1 Diferenciální rovnice 1. řádu

1.1.1 Řešte rovnice

1. $(xy^2 + x)dx + (y - x^2y)dy = 0$ $\left[1 + y^2 = C(1 - x^2)\right]$
2. $xyy' = 1 - x^2$ $\left[x^2 + y^2 = \ln Cx^2\right]$
3. $y' \operatorname{tg} x - y = a$ $[y = C \sin x - a]$
4. $xydx + (x + 1)dy = 0$ $[y = C(x + 1)e^{-x}]$
5. $\sqrt{y^2 + 1}dx = xydy$ $\left[\ln|x| = C + \sqrt{y^2 + 1}; x = 0\right]$
6. $e^y(1 + x^2)dy - 2x(1 + e^y)dx = 0$ $\left[1 + e^y = C(1 + x^2)\right]$
7. $(x^2 - 1)y' + 2xy^2 = 0; y(0) = 1$ $\left[y\{\ln(1 - x^2) + 1\} = 1\right]$
8. $y' \sin x = y \ln y; y(\frac{\pi}{2}) = e$ $\left[y = e^{\operatorname{tg}\frac{x}{2}}\right]$
9. $\sin y \cos x dy = \cos y \sin x dx; y(0) = \frac{\pi}{4}$ $\left[\cos x = \sqrt{2}\cos y\right]$
10. $y' \operatorname{ctg} x + y = 2; y(\frac{\pi}{3}) = 0$ $[y = 2 - 4\cos x]$

1.1.2 Převedením na rovnici se separovatelnými proměnnými řešte

1. $y' - y = 2x - 3$ $[2x + y - 1 = Ce^x]$
2. $y' = \sin(x - y)$ $\left[x + C = \operatorname{ctg}(\frac{y-x}{2} + \frac{\pi}{4})\right]$
3. $y' = \sqrt{4x + 2y - 1}$ $\left[\sqrt{4x + 2y - 1} - 2\ln(\sqrt{4x + 2y - 1} + 2) = x + C\right]$
4. $y' = \cos(x - y - 1)$ $\left[y = x - 1 - 2\operatorname{arctg}(\frac{1}{C-x}) + 2k\pi; k \in Z\right]$
5. $y'\sqrt{1 + x + y} = x + y - 1$ $\left[x + C = 2u + \frac{2}{3}\ln|u - 1| - \frac{8}{3}\ln(u + 2)\right]$ $\left[u = \sqrt{1 + x + y}\right]$

1.1.3 Řešte rovnice

1. $y' = \frac{x+y}{x-y}$ $\left[\operatorname{arctg}\frac{y}{x} = \ln C\sqrt{x^2 + y^2}\right]$
2. $y' = \frac{2xy}{x^2-y^2}$ $\left[x^2 + y^2 = Cy\right]$
3. $xy' - y = \sqrt{x^2 + y^2}$ $\left[x^2 = C^2 + 2Cy)\right]$
4. $(3y^2 + 3xy + x^)dx = (x^2 + 2xy)dy$ $\left[(x + y)^2 = Cx^3e^{-\frac{x}{x+y}}\right]$
5. $(x^2 + y^2)y' = 2xy$ $\left[y^2 - x^2 = Cy; y = 0\right]$

6. $xy' = y\cos\ln\frac{y}{x}$ $\left[\ln Cx = \operatorname{ctg}(\frac{1}{2}\ln\frac{y}{x})\right]$ $\left[y = xe^{2k\pi}, k\in Z\right]$

7. $y + \sqrt{x^2+y^2} - xy' = 0; y(1) = 0$ $\left[y = \frac{x^2-1}{2}\right]$

8. $(xy' - y)\operatorname{arctg}\frac{y}{x} = x; y(1) = 0$ $\left[\sqrt{x^2+y^2} = e^{\frac{y}{x}\operatorname{arctg}\frac{y}{x}}\right]$

9. $(y^2 - 3x^2)dy + 2xydx = 0; y(0) = 1$ $\left[y^3 = y^2 - x^2\right]$

10. $y' = \frac{y^2-2xy-x^2}{y^2+2xy-x^2}; y(1) = 1$ $[y = -x]$

1.1.4 Převedením na homogenní rovnici řešte

1. $(2x - 4y + 6)dx + (x + y - 3)dy = 0$ $\left[(y-2x)^3 = C(y-x-1)^2\right]$

2. $(2x + y + 1)dx - (4x + 2y - 3)dy = 0$ $\left[2x + y - 1 = Ce^{2y-x}\right]$

3. $(x + 4y)y' = 2x + 3y - 5$ $\left[(y - x + 5)^5(x + 2y - 2) = C\right]$

4. $y' = 2(\frac{y+2}{x+y-1})^2$ $\left[y + 2 = Ce^{-2\operatorname{arctg}\frac{y+2}{x-3}}\right]$

5. $(x + y + 1)dx + (2x + 2y - 1)dy = 0$ $[x + 2y + 3\ln|x + y - 2| = C]$

1.1.5 Řešte rovnice - variací konstant

1. $xy' - 2y = 2x^4$ $\left[y = Cx^2 + x^4\right]$

2. $xy' + y + 1 = 0$ $[xy = C - \ln|x|]$

3. $xy' + (x + 1)y = 3x^2e^{-x}$ $\left[xy = (x^3 + C)e^{-x}\right]$

4. $(xy + e^x)dx - xdy = 0$ $[y = e^x(\ln|x| + C)]$

5. $y = x(y' - x\cos x)$ $[y = x(C + \sin x)]$

6. $(xy' - 1)\ln x = 2y$ $\left[y = C\ln^2 x - \ln x\right]$

7. $y\sin x + y'\cos x = 1$ $[y = \sin x + C\cos x]$

8. $(2e^y - x)y' = 1$ $[x = e^y + Ce^{-y}]$

9. $y' = \frac{y}{3x-y^2}$ $\left[x = Cy^3 + y^2\right]$

10. $y' = \frac{1}{x\sin y + 2\sin 2y}$ $\left[x = 8\sin^2\frac{y}{2} + Ce^{-\cos y}\right]$

11. $y' + \frac{3y}{x} = \frac{2}{x^{-3}}; y(1) = 1$ $\left[y = -\frac{1}{x^3} + \frac{2}{x^2}\right]$

12. $y' - 2xy = 1; y(0) = 0$ $\left[y = e^{x^2}\int\limits_0^x e^{-t^2}dt\right]$

13. $2\sqrt{x}y' - y = -\sin\sqrt{x} - \cos\sqrt{x}$; y je omezená pro $\to \infty$ $\quad [y = \cos\sqrt{x}]$

14. $2x^2y' - xy = 2x\cos x - 3\sin x$; $y \to 0$ pro $x \to \infty$ $\quad \left[y = \frac{\sin x}{x}\right]$

15. $(1+x^2)\ln(1+x^2)y' - 2xy = \ln(1+x^2) - 2x \operatorname{arctg} x$ $\quad [y = \operatorname{arctg} x]$
$\left[y \to -\frac{\pi}{2} \text{ pro } x \to -\infty\right]$

1.1.6 Převedením na lineární rovnici řešte

1. $y' + 2y = y^2e^x$ $\quad \left[y(e^x + Ce^{2x}) = 1, y = 0\right]$

2. $xy' - 2x^2\sqrt{y} = 4y$ $\quad \left[y = x^4\ln^2 Cx, y = 0\right]$

3. $xy' + 2y + x^5y^3e^x = 0$ $\quad \left[y^{-2} = x^4(2e^x + C), y = 0\right]$

4. $(1+x^2)y' = xy + x^2y^2$ $\quad \left[\frac{1}{y} = \frac{1}{\sqrt{1+x^2}}(C - \frac{x}{2}\sqrt{1+x^2} - \frac{1}{2}\ln(x + \sqrt{x^2+1}))\right]$

1.2 Lineární difernicální rovnice n-tého řádu

1.2.1 Rozhodněte o lineární závislosti nebo nezávislosti systémů funkcí

1. $4, x$ $\quad$ [nezávislé]

2. $1, 2, x, x^2$ $\quad$ [závislé]

3. e^x, xe^x, x^2e^x $\quad$ [nezávislé]

4. $5, \cos^2 x, \sin^2 x$ $\quad$ [závislé]

5. $\cos x, \cos(x+1), \cos(x-2)$ $\quad$ [závislé]

6. $1, \arcsin x, \arccos x$ $\quad$ [závislé]

7. $\cos x, \sin x, \cos 2x$ $\quad$ [nezávislé]

1.2.2 Najděte Wronskián funkcí

1. $1, x$ $\quad [1]$

2. e^{-x}, xe^{-x} $\quad \left[e^{-2x}\right]$

3. $2, \cos x, \cos 2x$ $\quad \left[-8\sin^3 x\right]$

4. $4, \sin^2 x, \cos 2x$ $\quad [0]$

5. $e^{-3x}\sin 2x, e^{-3x}\cos 2x$ $\quad \left[-2e^{-6x}\right]$

1.2.3 Nalezněte obecné řešení následujících rovnic, jestliže znáte partikulární integrál

1. $(\sin x - \cos x)y'' - 2\sin(x)y' + (\cos x + \sin x)y = 0$ $[y = C_1e^x + C_2 \sin x, y_1 = e^x]$
2. $(1 - x^2)y'' - xy' + \frac{1}{4}y = 0; y_1 = \sqrt{1+x}$ $\left[y = C_1\sqrt{1+x} + C_2\sqrt{1-x}\right]$
3. $x^2(x+1)y'' - 2y = 0; y_1 = 1 + \frac{1}{x}$ $\left[y = C_1(1 + \frac{1}{x}) + C_2(\frac{x}{2} + 1 - \frac{x+1}{x}\ln|x+1|)\right]$
4. $xy'' + 2y' - xy = 0; y_1 = \frac{e^x}{x}$ $[xy = C_1e^{-x} + C_2e^x]$
5. $y'' - 2(1 + \operatorname{tg}^2 x)y = 0; y_1 = \operatorname{tg} x$ $[y = C_1 \operatorname{tg} x + C_2(1 + x\operatorname{tg} x)]$
6. $(e^x + 1)y'' - 2y' - e^xy = 0; y_1 = e^x - 1$ $\left[y = C_1(e^x - 1) + \frac{C_2}{e^x+1}\right]$
7. $x^2(2x-1)y''' + (4x-3)xy'' - 2xy' + 2y = 0$ $\left[y = C_1x + \frac{C_2}{x} + C_3(x\ln|x| + 1)\right]$ $\left[y_1 = x, y_2 = \frac{1}{x}\right]$
8. $(x^2 - 2x + 3)y''' - (x^2 + 1)y'' + 2xy' - 2y = 0$ $\left[y = C_1x + C_2e^x + C_3(x^2 - 1)\right]$ $[y_1 = x, y_2 = e^x]$

1.2.4 Najděte obecné řešení rovnic (hledejte partikulární integrál ve tvaru $y = e^{\lambda x}$ nebo ve tvaru polynomu)

1. $(2x+1)y'' + 4xy' - 4y = 0$ $\left[y = C_1x + C_2e^{-2x}\right]$
2. $xy'' - (2x+1)y' + (x+1)y = 0$ $\left[y = (C_1x^2 + C_2)e^x\right]$
3. $x(x-1)y'' - xy' + y = 0$ $[y = C_1(1 + x\ln|x|) + C_2x]$
4. $(x^2 - 1)y'' + (x-3)y' - y = 0$ $\left[y = C_1(x-3) + \frac{C_2}{x+1}\right]$
5. $xy'' - (x+1)y' - 2(x-1)y = 0$ $\left[y = C_1e^{2x} + C_2(3x+1)e^{-x}\right]$

1.2.5 Řešte rovnice

1. $3y'' - 2y' - 8y = 0$ $\left[y = C_1e^{2x} + C_2e^{-\frac{4}{3}x}\right]$
2. $y''' - 3y'' + 3y' - y = 0$ $[y = e^x(1+x), y(0) = 1, y'(0) = 2, y''(0) = 3]$
3. $y'' - 4y' + 3y = 0; y(0) = 6, y'(0) = 10$ $\left[y = 4e^x + 2e^{3x}\right]$
4. $y''' + 6y'' + 11y' + 6y = 0$ $\left[y = C_1e^{-x} + C_2e^{-2x} + C_3e^{-3x}\right]$
5. $y^{(6)} + 2y^{(5)} + y^{(4)} = 0$ $\left[y = C_1 + C_2x + C_3x^2 + C_4x^3 + e^{-x}(C_5 + C_6x)\right]$
6. $4y'' - 8y' + 5y = 0$ $\left[y = e^x(C_1\cos\frac{x}{2} + C_2\sin\frac{x}{2})\right]$
7. $y''' - 8y = 0$ $\left[y = C_1e^{2x} + e^{-x}(C_2\cos\sqrt{3}x + C_3\sin\sqrt{3}x\right]$
8. $y^{(4)} + 4y''' + 10y'' + 12y' + 5y = 0$ $[y = (C_1 + C_2x)e^{-x} + (C_3\cos 2x + C_4\sin 2x)e^{-x}]$
9. $y'' - 2y' + 2y = 0; y(0) = 0, y'(0) = 1$ $[y = e^x\sin x]$
10. $y'' - 2y' + 3y = 0; y(0) = 1, y'(0) = 3$ $\left[y = e^x(\cos\sqrt{2}x + \sqrt{2}\sin\sqrt{2}x\right]$

1.2.6 Sestavte lineární homogenní diferenciální rovnici s konstantními koeficienty, je-li její fundamentální systém

1. e^{-x}, e^{x} $[y'' - y = 0]$
2. $1, e^{x}$ $[y'' - y' = 0]$
3. $\sin 3x, \cos 3x$ $[y'' + 9y = 0]$
4. e^{-2x}, xe^{-2x} $[y'' + 4y' + 4y = 0]$
5. e^{x}, xe^{x}, e^{2x} $[y''' - 4y'' + 5y' - 2y = 0]$
6. $1, e^{-x}\sin x, e^{-x}\cos x$ $[y'' + 2y'' + 2y' = 0]$

1.2.7 Napište lineární homogenní diferenciální rovnici s konstantními koeficienty nejmenšího možného řádu tak, aby měla řešení

1. $y_1 = x^2 e^{x}$ $[y''' - 3y'' + 3y' - y = 0]$
2. $y_1 = e^{2x}\cos x$ $[y'' - 4y' + 5y = 0]$
3. $y_1 = x \sin x$ $\left[y^{(4)} + 2y'' + y = 0\right]$
4. $y_1 = x, y_2 = \sin x$ $\left[y^{(4)} + y'' = 0\right]$
5. $y_1 = xe^{x}\cos 2x$ $\left[y^{(4)} - 4y''' + 14y'' - 20y' + 25y = 0\right]$

1.2.8 Řešte rovnice - variací konstant

1. $y'' - 2y' + y = \frac{e^x}{x}$ $[y = e^{x}(x \ln|x| + C_1 x + C_2)]$
2. $y'' - 2y' + y = \frac{e^x}{x^2+1}$ $\left[y = e^{x}(C_1 x + C_2 - \ln\sqrt{x^2+1} + x \operatorname{arctg} x)\right]$
3. $y'' + 3y' + 2y = \frac{1}{e^x+1}$ $\left[y = (e^{-x} + e^{-2x})\ln(e^{x}+1) + C_1 e^{-x} + C_2 e^{-2x}\right]$
4. $y'' + y + \operatorname{ctg}^2 x = 0$ $\left[y = 2 + C_1 \cos x + C_2 \sin x + \cos(x)\ln|\operatorname{tg}\frac{x}{2}|\right]$

$d'' - y' = f(x)$

5. $f(x) = \frac{e^x}{1+e^x}$ $[y = e^{x}(x + C_1) - (e^{x}+1)\ln(e^{x}+1) + C_2]$
6. $f(x) = e^{2x}\sqrt{1-e^{2x}}$ $\left[y = \frac{1}{2}e^{x}(\arcsin(e^{x}) + e^{x}\sqrt{1-e^{2x}} + C_1) + \frac{1}{3}\sqrt{(1-e^{2x})^3} + C_2\right]$
7. $f(x) = e^{2x}\cos(e^{x})$ $[y = C_1 e^{x} - \cos(e^{x}) + C_2]$

1.2.9 Řešte rovnice - odhadem y_p

1. $y'' + y = 4xe^x$ $\quad [y = C_1 \cos x + C_2 \sin x + (2x+2)e^x]$
2. $y'' - y = 2e^x - x^2$ $\quad [y = C_1 e^x + C_2 e^{-x} + xe^x + x^2 + 2]$
3. $y'' + y' - 2y = 3xe^x$ $\quad \left[y = C_1 e^x + C_2 e^{-2x} + (\frac{x^2}{2} - \frac{x}{3})e^x\right]$
4. $y'' - 3y' + 2y = \sin x$ $\quad [y = C_1 e^x + C_2 e^{2x} + \frac{\sin x}{10} + \frac{3\cos x}{10}]$
5. $y'' + y = 4 \sin x$ $\quad [y = C_1 \cos x + C_2 \sin x - 2x \cos x]$
6. $y'' - 3y' + 2y = x \cos x$ $\quad [y = C_1 e^x + C_2 e^{2x} + (\frac{x}{10} - \frac{12}{100}) \cos x - (\frac{3x}{10} + \frac{34}{100}) \sin x]$
7. $y'' + 3y' - 4y = e^{-4x} + xe^{-x}$ $\quad [y = C_1 e^x + C_2 e^{-4x} - \frac{x}{5} e^{-4x} - (\frac{x}{6} + \frac{1}{36})e^{-x}]$
8. $y'' - 9y = e^{3x} \cos x$ $\quad [y = C_1 e^{3x} + C_2 e^{-3x} + e^{3x}(\frac{6}{37} \sin x - \frac{1}{37} \cos x)]$
9. $y'' - 2y' + y = 6xe^x$ $\quad [y = (C_1 + C_2 x + x^3)e^x]$
10. $y'' + y = x \sin x$ $\quad \left[y = (C_1 - \frac{x^2}{4}) \cos x + (C_2 + \frac{x}{4}) \sin x\right]$

1.2.10 Řešte rovnice s počáteční podmínkou - odhadem y_p

1. $y'' + 9y = 6e^{3x}; y(0) = y'(0) = 0$ $\quad [y = -\frac{1}{3}(\cos 3x + \sin 3x - e^{3x})]$
2. $y'' - 4y' + 5y = 2x^2 e^x; y(0) = 2, y'(0) = 3$ $\quad [y = e^{2x}(\cos x - 2 \sin x) + (x+1)^2 e^x]$
3. $y'' + 6y' + 9y = 10 \sin x; y(0) = y'(0) = 0$ $\quad [y = (x + \frac{3}{5})e^{-3x} + \frac{1}{5}(4 \sin x - 3 \cos x)]$
4. $y'' + 4y = \sin x; y(0) = y'(0) = 1$ $\quad [y = \cos 2x + \frac{1}{3}(\sin 2x + \sin x)]$
5. $y'' + y = 2 \cos x; y(0) = 1, y'(0) = 0$ $\quad [y = \cos x + x \sin x]$

1.2.11 Odhadněte partikulární integrály následujících rovnic

1. $y'' - 7y' = (x-1)^2$ $\quad [A_1 x^3 + A_2 x^2 + A_3 x]$
2. $y'' + 7x' = e^{-7x}$ $\quad [Axe^{-7x}]$
3. $y'' - 8y' + 16y = (10 - x)e^{4x}$ $\quad [(A_1 x^3 + A_2 x^2)e^{4x}]$
4. $y'' + 25y = \cos 5x$ $\quad [x(A \cos 5x + B \sin 5x)]$
5. $y'' + 4y' + 8y = e^{2x}(\sin 2x + \cos 2x)$ $\quad [(A \cos 2x + B \sin 2x)e^{2x}]$
6. $y'' - 4y' + 8y = e^{2x}(\sin 2x - \cos 2x)$ $\quad [x(A \cos 2x + B \sin 2x)e^{2x}]$
7. $y'' + 4y = \sin x \sin 2x$ $\quad [A_1 \cos x + B_1 \sin x +]$ $[+A_2 \cos 3x + B_2 \sin 3x]$
8. $y^{(4)} - y''' = 4$ $\quad [Ax^3]$

9. $y''' + 2y'' + y' = (2x+1)\sin x + (x^2 - 4x)\cos x$ $\left[(Ax^2 + Bx + C)\cos x + \right]$ $\left[+(Dx^2 + Ex + F)\sin x\right]$

10. $y''' - y' = e^x \sin x + 2x^2$ $[e^x(A\cos x + B\sin x) +]$ $\left[+x(Cx^2 + Dx + E)\right]$

11. $y^{(4)} - 4y''' + 8y'' - 8y' + 4y = e^x(x\cos x + \sin x)$ $\left[x^2 e^x\{(Ax+B)\cos x + \right]$ $[+(Cx+D)\sin x\}]$

12. $y^{(5)} - y^{(4)} + 8y''' - 8y'' + 16y' - 16y = 3\cos 2x + 1$ $\left[x^2(A\cos 2x + B\sin 2x) + C\right]$ $[y = 3\cos 2x + 1]$

1.2.12 Vypočtěte partikulární integrál rovnice $y'' - 3y' + 2y = f(x)]$, je-li $f(x)$ rovno

1. $10e^{-x}$ $\left[\frac{5}{3}e^{-x}\right]$
2. $3e^{2x}$ $\left[3xe^{2x}\right]$
3. $2\sin x$ $\left[\frac{3}{5}\cos x + \frac{1}{5}\sin x\right]$
4. $2x^3 - 30$ $\left[x^3 + \frac{9}{2}x^2 + \frac{21}{2}x - \frac{15}{4}\right]$
5. $2e^x \cos\frac{x}{2}$ $\left[-\frac{8}{5}e^x(\cos\frac{x}{2} + 2\sin\frac{x}{2})\right]$
6. $x - e^{-2x} + 1$ $\left[\frac{x}{2} + \frac{5}{4} - \frac{1}{12}e^{-2x}\right]$
7. $e^x(3 - 4x)$ $\left[e^x(2x^2 + x)\right]$
8. $3x + 5\sin 2x$ $\left[\frac{3}{2}x + \frac{1}{4}(9 + 3\cos 2x - \sin 2x)\right]$
9. $2e^x - e^{-2x}$ $\left[-2xe^x - \frac{1}{12}e^{-2x}\right]$
10. $\sin x \sin 2x$ $\left[\frac{1}{20}\cos x - \frac{3}{20}\sin x + \frac{7}{260}\cos 3x + \frac{9}{260}\sin 2x\right]$
11. $\sinh x$ $\left[-\frac{1}{12}e^{-x} - \frac{1}{2}xe^x\right]$

1.2.13 Vypočtěte partikulární integrál rovnice $2y'' + 5y' = f(x)$, je-li $f(x)$ rovno

1. $5x^2 - 2x - 1$ $\left[\frac{1}{3}x^3 - \frac{3}{5}x^2 + \frac{7}{25}x\right]$
2. $100xe^{-x}\cos x$ $\left[\frac{(650x+2650)\sin x - (3250x-400)\cos x}{169e^x}\right]$
3. $29\cos x$ $[5\sin x - 2\cos x]$
4. $\cos^2 x$ $\left[\frac{x}{10} + \frac{5}{164}\sin 2x - \frac{1}{41}\cos 2x\right]$
5. $3\cosh\frac{5}{2}x$ $\left[\frac{3}{10}(\frac{1}{5}e^{\frac{5}{2}x} - xe^{-\frac{5}{2}x})\right]$
6. $29x\sin x$ $\left[(-5x - \frac{16}{29})\cos x - (2x - \frac{185}{29})\sin x\right]$
7. e^x $\left[\frac{1}{7}e^x\right]$
8. $\frac{1}{10}e^{-\frac{25x}{10}} - 25\sin\frac{25x}{10}$ $\left[\cos\frac{25x}{10} + \sin\frac{25x}{10} - \frac{2x}{100}e^{-\frac{25x}{10}}\right]$

1.2.14 Vypočtěte partikulární integrál rovnice $y'' - 4y' + 4y = f(x)$, je-li $f(x)$ rovno

1. 1 $\left[\frac{1}{4}\right]$
2. e^{-x} $\left[\frac{1}{9}e^{x}\right]$
3. $3e^{2x}$ $\left[\frac{3}{2}x^2e^{2x}\right]$
4. $2(\sin 2x + x)$ $\left[\frac{1}{4}\cos 2x + \frac{1}{2}x + \frac{1}{2}\right]$
5. $\sin x \cos 2x$ $\left[\frac{1}{169}(-\frac{5}{2}\sin 3x + 6\cos 3x) + \frac{1}{50}(3\sin x + 4\cos x)\right]$
6. $\sin^3 x$ $\left[\frac{3}{100}(3\sin x + 4\cos x) + \frac{1}{676}(5\sin 3x - 12\cos 3x)\right]$
7. $8(x^2 + e^{2x} + \sin 2x)$ $\left[2x^2 + 4x + 3 + 4x^2e^{2x} + \cos 2x\right]$
8. $\sinh 2x$ $\left[\frac{1}{4}(x^2e^{2x} - \frac{1}{8}e^{-2x})\right]$
9. $\sinh x + \sin 2x$ $\left[\frac{1}{2}(e^{x} - \frac{1}{9}e^{-x}) + \frac{1}{25}(3\sin x + 4\cos x)\right]$
10. $e^{x} - \sinh(x-1)$ $\left[e^{x} - \frac{1}{2}e^{x-1} + \frac{1}{18}e^{1-x}\right]$

1.2.15 Vypočtěte partikulární integrál rovnice $5y'' - 6y' + 5y = f(x)$, je-li $f(x)$ rovno

1. $5e^{\frac{3}{5}x}$ $\left[\frac{25}{16}e^{\frac{3}{5}x}\right]$
2. $\sin \frac{4}{5}x$ $\left[\frac{15}{219}\sin\frac{4}{5}x + \frac{40}{219}\cos\frac{4}{5}x\right]$
3. $e^{2x} + 2x^3 - x + 2$ $\left[\frac{1}{13}e^{2x} + \frac{1}{5}(2x^3 + \frac{36}{5}x^2 + \frac{107}{25}x - \frac{1118}{125})\right]$
4. $e^{\frac{3}{5}x}\cos x$ $\left[\frac{5}{9}e^{\frac{3}{5}x}\cos x\right]$
5. $e^{\frac{3}{5}x}\sin\frac{4}{5}x$ $\left[-\frac{1}{8}xe^{\frac{3}{5}x}\cos\frac{4}{5}x\right]$
6. $e^{x}\cosh x$ $\left[\frac{1}{26}e^{2x} + \frac{1}{10}\right]$

1.2.16 Řešte Eulerovo rovnice, substituce $x = e^t$ nebo řešení tvaru $y = x^\lambda$

1. $x^2y'' - 3xy' + 3y = 0$ $\left[y = C_1x + C_2x^3\right]$
2. $xy''' + y'' = 0$ $[y = C_1 + x(C_2 + C_3\ln x]$
3. $x^3y''' - 3x^2y'' + 6xy' - 6y = x$ $\left[y = C_1x + C_2x^2 + C_3x^3 + \frac{1}{2}x\ln x\right]$
4. $x^2y'' - xy' + y = 6x\ln x$ $\left[y = x\ln^3 x + x(C_1 + C_2\ln x)\right]$
5. $x^2y'' + xy' + y = 2\sin(\ln x)$ $[y = -\ln x\cos(\ln x) + C_1\cos(\ln x) + C_2\sin(\ln x)]$

1.2.17 Řešte následující okrajové úlohy

1. $y'' - y = 0; y(0) = 0, y(2\pi) = 1$ $\left[y = \frac{\sinh x}{\sinh 2\pi}\right]$
2. $y'' + y = 0; y(0) = 0, y(2\pi) = 1$ [nemá řešení]
3. $y'' - k^2 y = 0; y(0) = v_1, y(x_0) = v_2$ $\left[y = \frac{1}{\sinh kx_0}(v_1 \sinh k(x_0 - x) + v_2 \sinh kx)\right]$
4. $y'' - \alpha^2 y = 0; y(0) = v, y'(x_0) = 0$ $\left[y = v\frac{\cosh(x_0 - x)}{\cosh \alpha x_0}\right]$
5. $y'' - \alpha^2 s y = 0; y(0) = \frac{1}{s}, y'(x_0) = 0$ $\left[s < 0; y = \frac{\cos\alpha\sqrt{-s}(x_0 - x)}{s\cos\alpha\sqrt{-s}x_0} \text{ pro } x_0 \neq \frac{(2k+1)\pi}{2\alpha\sqrt{-s}}\right]$
   $\left[\text{pro } x_0 = \frac{(2k+1)\pi}{2\alpha\sqrt{-s}} \text{nemá řešení; } s > 0; y = \frac{\cosh\alpha\sqrt{s}(x_0 - x)}{s\cosh\alpha\sqrt{s}x_0}; k = 1, 2, 3, ...\right]$
6. $y'' - \lambda^2 y = 0; \lambda \neq 0, y(0) = 0, y(1) = \frac{1}{\lambda}$ $\left[y = \frac{\sinh \lambda x}{\lambda \sinh \lambda}\right]$
7. $y'' - \lambda^2 y = 0; \lambda \neq, y(0) = 0, y'(1) = \frac{1}{\lambda}$ $\left[y = \frac{\sinh \lambda x}{\lambda^2 \cosh \lambda}\right]$
8. $y'' - \lambda^2 y = 0; \lambda \neq, y'(0) = 0, y(1) = \frac{1}{\lambda}$ $\left[y = \frac{\cosh \lambda x}{\lambda \cosh \lambda}\right]$
9. $y'' - \alpha^2 s^2 y = \alpha^2 gl; y(0) = y(x_0) = 0$ tady jeden příklad chybí je na konci stránky
10. $xy'' + y' = 0; y(1) = \alpha y'(1); y(x)$ je omezená pro $x \to \infty$ $[y = 0]$
11. $y^{(4)} - \lambda^4 y = 0; y(0) = y''(0) = 0, y(\pi) = y''(\pi) = 0$ $[y = C \sin kx \text{ pro } \lambda = k]$
    $[k = 1, 2, 3, ... \quad y = 0 \text{ pro ostatní } \lambda]$

1.2.18 Najděte vlastní čísla a vlastní funkce úlohy $y'' + \lambda y = 0$, je-li

1. $x \in< 0, \pi >, y(0) = y(\pi) = 0$ $[\lambda_K = K^2, y_K = \sin Kx, K \in N]$
2. $x \in< 0, \pi >, y(0) = y'(\pi) = 0$ $\left[\lambda_K = \frac{(2K-1)^2}{4}, y_K = \sin\frac{2K-1}{2}x, K \in N\right]$
3. $x \in< 0, \pi >, y'(0) = y(\pi) = 0$ $\left[\lambda_K = \frac{(2K-1)^2}{4}, y_K = \cos\frac{2K-1}{2}x, K \in N\right]$
4. $x \in< 0, \pi >, y'(0) = y'(\pi) = 0$ $[\lambda_K = K^2, y_K = \cos Kx, K = 0, 1, 2, \ldots]$
5. $x \in< 1, 2 >, y(1) = y(2) = 0$ $[\lambda_K = K^2\pi^2, y_K = \sin K\pi x, K \in N]$
6. $x \in< 1, 2 >, y(1) = y'(2) = 0$ $\left[\lambda_K = \frac{(2K-1)^2\pi^2}{4}, y_K = \cos\frac{2K-1}{2}\pi x, K \in N\right]$
7. $x \in< 1, 2 >, y'(1) = y(2) = 0$ $\left[\lambda_K = \frac{(2K-1)^2\pi^2}{4}, y_K = \sin\frac{2k-1}{2}\pi x, K \in N\right]$
8. $x \in< 1, 2 >, y'(1) = y'(2) = 0$ $[\lambda_K = K^2\pi^2, y_K = \cos K\pi x; K = 0, 1, 2, \ldots]$
9. $x \in< a, b >, y(a) = y(b) = 0$ $\left[\lambda_K = \frac{K^2\pi^2}{(b-a)^2}, y_K = \sin\frac{K\pi(x-a)}{b-a}, K \in N\right]$
10. $x \in< a, b >, y(a) = y'(b) = 0$ $\left[\lambda_K = \frac{(2K-1)^2\pi^2}{4(b-a)^2}, y_K = \sin\frac{(2K-1)(x-a)\pi}{2(b-a)}, K \in N\right]$

11. $x \in < a, b >, y'(a) = y(b) = 0$ $\left[\lambda_K = \frac{(2K-1)^2\pi^2}{4(b-a)^2}, y_K = \cos\frac{(2K-1)(x-a)\pi}{2(b-a)}, K \in N\right]$

12. $x \in < a, b >, y'(a) = y'(b) = 0$ $\left[\lambda_K = \frac{K^2\pi^2}{(b-a)^2}, y_K = \cos\frac{(x-a)K\pi}{b-a}, K = 0, 1, 2, \dots\right]$

1.2.19 Najděte vlastní čísla a vlastní funkce následujících okrajových úloh

1. $y'' + 2y' + \lambda y = 0; x \in < 0, l >, y(0) = y(l) = 0$ $\left[\lambda_K = 1 + \frac{K^2\pi^2}{\ln^2 l}\right]$
   $\left[y_K = l^{-x}\sin\frac{K\pi x}{l}, K \in N\right]$

2. $x^2y'' + xy' + \lambda y = 0; x \in < 1, l >, y(1) = y(l) = 0$ $\left[\lambda_K = \frac{K^2\pi^2}{\ln^2 l}, y_K = \sin\frac{K\pi\ln x}{\ln l}\right]$

3. $y'' + (\lambda + 1)y = 0$ $\left[\lambda_K = K^2\pi^2 - 1, K \in N\right]$
   $x \in < 0, 1 >, y(0) = y'(0) = 0, y(1) - y'(1) = 0$ $[y_K = \sin(\mathrm{arctg}(K\pi) + K\pi x)]$

4. $y'' + \frac{2}{x}y' + \lambda y = 0; y(l) = 0, y$ je omezená pro $x \to 0$ $\left[\lambda_K = \frac{K^2\pi^2}{l^2}, y_K = \frac{1}{x}\sin\frac{K\pi x}{l}\right]$
   převeďte danou rovnici na ??????? tvar

1.2.20 Rozhodněte, zda jsou nezávislé prvé integrály $C_1 = \frac{x+y}{z+x}, C_2 = \frac{z-y}{x+y}$ soustavy $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

1.3 Soustavy lineárních diferenciálních rovnic

1.3.1 Řešte následující soustavy homogenních diferenciálních rovnic

1. $x' = 2x + y$ $\left[x = C_1e^t + C_2e^{5t}\right]$
   $y' = 3x + 4y$ $\left[y = -C_1e^t + 3C_2e^{5t}\right]$

2. $x' = x - y$ $\left[x = C_1e^{-t} + C_2e^{3t}\right]$
   $y' = y - 4x$ $\left[y = 2C_1e^{-t} - 2C_2e^{3t}\right]$

3. $x' + x - 8y = 0$ $\left[x = 2C_1e^{3t} - 4C_2e^{-3t}\right]$
   $y' - x - y = 0$ $\left[y = C_1e^{3t} + C_2e^{-3t}\right]$

4. $x' = x + y$ $\left[x = e^{2t}(C_1\cos t + C_2\sin t\right]$
   $y' = 3y - 2x$ $\left[y = e^{2t}\{(C_1 + C_2)\cos t + (C_2 - C_1)\sin t\}\right]$

5. $x' = x - 3y$ $[x = e^t(C_1\cos 3t + C_2\sin 3t)]$
   $y' = 3x + y$ $[y = e^t(C_1\sin 3t - C_2\cos 3t)]$

6. $x' + x + 5y = 0$ $[x = (2C_2 - C_1)\cos 2t - (2C_1 + C_2)\sin 2t]$
   $y' - x - y = 0$ $[y = C_1\cos 2t + C_2\sin 2t]$

7. $x' = 2x + y$ $\left[x = (C_1 + C_2t)e^{3t}\right]$
   $y' = 4y - x$ $\left[y = (C_1 + C_2 + C_2t)e^{3t}\right]$

8. $x' = 3x - y$ $[x = (C_1 + C_2t)e^t]$
   $y' = 4x - y$ $[y = (2C_1 - C_2 + 2C_2t)e^t]$

9. $x' = x + z - y$ $\quad$ $[x = C_1e^t + C_2e^{2t} + C_3e^{-t}]$
$y' = x + y - z$ $\quad$ $[y = C_1e^t - 3C_3e^{-t}]$
$z' = 2x - y$ $\quad$ $[z = C_1e^t + C_2e^{2t} - 5C_3e^{-t}]$

10. $x' = 3x - y + z$ $\quad$ $[x = C_1e^t + C_2e^{2t} + C_3e^{5t}]$
$y' = x + y + z$ $\quad$ $[y = C_1e^t - 2C_2e^{2t} + C_3e^{5t}]$
$z' = 4x - y + 4z$ $\quad$ $[z = -C_1e^t - 3C_2e^{2t} + 3C_3e^{5t}]$

11. $x' = 4y - 2z - 3x$ $\quad$ $[x = C_1e^t + C_3e^{-t}]$
$y' = z + x$ $\quad$ $[y = C_1e^t + C_2e^{2t}]$
$z' = 6x - 6y + 5z$ $\quad$ $[z = 2C_2e^{2t} - C_3e^{-t}]$

12. $x' = x - y - z$ $\quad$ $[x = e^t(2C_2 \sin 2t + 2C_3 \cos 2t)]$
$y' = x + y$ $\quad$ $[y = e^t(C_1 - C_2 \cos 2t + C_3 \sin 2t)]$
$z' = 3x + z$ $\quad$ $[z = e^t(-C_1 - 3C_3 \cos 2t + 3C_3 \sin 2t]$

13. $x' = 4x - y - z$ $\quad$ $[x = C_1e^{2t} + (C_2 + C_3)e^{3t}]$
$y' = x + 2y - z$ $\quad$ $[y = C_1e^{2t} + C_2e^{3t}]$
$z' = x - y + 2z$ $\quad$ $[z = C_1e^{2t} + C_3e^{3t}]$

14. $x' = x - y + z$ $\quad$ $[x = (C_1 + C_2t)e^t + C_3e^{2t}]$
$y' = x + y - z$ $\quad$ $[y = (C_1 - 2C_2 + C_2t)e^t]$
$z' = 2z - y$ $\quad$ $[z = (C_1 - C_2 + C_2t)e^t + C_3e^{2t}]$

15. $x' = 4x - y$ $\quad$ $[x = (C_1 + C_2t + C_3t^2)e^{2t}]$
$y' = 3x + y - z$ $\quad$ $[y = \{2C_1 - C_2 + (2C_2 - 2C_3)t + 2C_3t^2\}e^{2t}]$
$z' = x + z$ $\quad$ $[z = \{C_1 - C_2 + 2C_3 + (C_2 - 2C_3)t + C_3t^2\}e^{2t}]$

1.3.2 Řešte následující soustavy nehomogenních diferenciálních rovnic

1. $x' = y + 2e^t$ $\quad$ $[x = C_1e^t + C_2e^{-t} + te^t - t^2 - 2]$
$y' + x + t^2$ $\quad$ $[y = C_1e^t - C_2e^{-t} + (t-1)e^t - 2t]$

2. $x' = y - 5\cos t$ $\quad$ $[x = C_1e^{2t} + C_2e^{-t} - 2\sin t - \cos t]$
$y' = 2x + y$ $\quad$ $[y = 2C_1e^{2t} - C_2e^{-t} + \sin t + 3\cos t]$

3. $x' = 4x + y - e^{2t}$ $\quad$ $[x = C_1e^{2t} + C_2e^{3t} + (t+1)e^{2t}]$
$y' = y - 2x$ $\quad$ $[y = -2C_1e^{2t} - C_2e^{3t} - 2te^{2t}]$

4. $x' = 2y - x + 1$ $\quad$ $[x = (C_1 + 2C_2t)e^t - 3]$
$y' = 3y - 2x$ $\quad$ $[y = (C_1 + C_2 + 2C_2t)e^t - 2]$

5. $x' = 5x - 3y + 2e^{3t}$ $\quad$ $[x = C_1e^{2t} + 3C_2e^{4t} - e^{-t} - 4e^{3t}]$
$y' = x + y + 5e^{-t}$ $\quad$ $[y = C_1e^{2t} + C_2e^{4t} - 2e^{-t} - 2e^{3t}]$

6. $x' = 2x - 4y$ $\quad$ $[x = 4C_1e^t + C_2e^{-2t} - 4te^t]$
$y' = x - 3y + 3e^t$ $\quad$ $[y = C_1e^t + C_2e^{-2t} - (t-1)e^t]$

7. $x' = 2x - y$ $\quad [x = C_1e^{3t} + 3t^2 + 2t + C_2]$
$y' = y - 2x + 18t$ $\quad [y = -C_1e^{3t} + 6t^2 - 2t + 2C_2 - 2]$

8. $x' = x + 2y + 16te^t$ $\quad [x = 2C_1e^{2t} + C_2e^{-3t} - (12t + 13)e^t]$
$y' = 2x - 2y$ $\quad [y = C_1e^{2t} - 2C_2e^{-3t} - (8t + 6)e^t]$

9. $x' = 2x - y$ $\quad [x = (C_1 + C_2t - t^2)e^t]$
$y' = x + 2e^t$ $\quad [y = \{C_1 - C_2 + (C_2 + 2)t - t^2\}e^t]$

10. $x' = x - y + 8t$ $\quad [x = C_1 \cos 2t - C_2 \sin 2t + 2t + 2]$
$y' = 5x - y$ $\quad [y = (C_1 + 2C_2) \cos 2t + (2C_1 - C_2) \sin 2t + 10t]$

11. $x' = 2x - y$ $\quad [x = C_1e^t + C_2e^{3t} + e^t(2 \cos t - \sin t)]$
$y' = 2y - x - 5e^t \sin t$ $\quad [y = C_1e^t - C_2e^{3t} + e^t(3 \cos t + \sin t)]$

12. $x' = y + \operatorname{tg}^2 t - 1$ $\quad [x = C_1 \cos t + C_2 \sin t + \operatorname{tg} t]$
$y' = -x + \operatorname{tg} t$ $\quad [y = -C_1 \sin t + C_2 \cos t + 2]$

13. $x' = -4x - 2y + \frac{2}{e^t-1}$ $\quad [x = C_1 + 2C_2e^{-t} + 2e^{-t} \ln |e^t - 1|]$
$y' = 6x + 3y - \frac{3}{e^t-1}$ $\quad [y = -2C_1 - 3C_2e^{-t} - 3e^{-t} \ln |e^t - 1|]$

14. $x' = x - y + \frac{1}{\cos t}$ $\quad [x = C_1 \cos t + C_2 \sin t + t(\cos t + \sin t) + (\cos t - \sin t) \ln | \cos t|]$
$y' = 2x - y$ $\quad [y = (C_1 - C_2) \cos t + (C_1 + C_2) \sin t + 2 \cos t \ln | \cos t| + 2t \sin t]$

15. $x' = 2x + y - 2z - t + 2$ $\quad [x = C_1e^t + C_2 \sin t + C_3 \cos t]$
$y' = 1 - x$ $\quad [y = t - C_1e^t + C_2 \cos t - C_3 \sin t]$
$z' = x + y - z - t + 1$ $\quad [z = 1 + C_2 \sin t + C_3 \cos t]$

1.3.3 Najděte partikulární řešení následujících soustav diferenciálních rovnic

1. $y' = y + z; y(0) = 0, z(0) = -1$ $\quad [y = e^{2t} - e^{3t}]$
$z' = -2y + 4z$ $\quad [z = e^{2t} - 2e^{3t}]$

2. $y' = 3y - z; y(0) = 1, z(0) = 5$ $\quad [y = e^{-2t}]$
$z' = 10y - 4z$ $\quad [z = 5e^{-2t}]$

3. $x' = 3x + 8y; x(0) = 6, y(0) = -2$ $\quad [x = 2(2e^t + e^{-t})]$
$y' = -3y - x$ $\quad [y = -e^t - e^{-t}]$

4. $x' = e^t - y - 5x; x(0) = \frac{119}{900}, y(0) = \frac{211}{900}$ $\quad [x = \frac{4}{25}e^t - \frac{1}{36}e^{2t}]$
$y' = e^{2t} + x - 3y$ $\quad [y = \frac{1}{25}e^t + \frac{7}{36}e^{2t}]$

5. $x' = y; x(0) = y(0) = 1$ $\quad [x = \cos t + \sin t]$
$y' = -x$ $\quad [y = \cos t - \sin t]$

6. $x' = 4x - 5y; x(0) = 0, y(0) = 1$ $\quad [x = (1 - 2t)e^{-2t}]$
$y' = x$ $\quad [y = te^{-2t}]$

7. $x' = x + y + t; x(0) = -\frac{7}{9}, y(0) = -\frac{5}{9}$ $\quad [x = -\frac{4}{3}t - \frac{7}{9}]$
$y' = x - 2y + 2t$ $\quad [y = \frac{1}{3}t - \frac{5}{9}]$

8. $x' = x + 5y; x(0) = -2, y(0) = 1$ $\quad [x = (\sin t - 2\cos t)e^{-t}]$
   $y' = -3y - x$ $\quad [y = e^{-t}\cos t]$

9. $2x' = 6x - y - 6t^2 - t + 3; x(0) = 2, y(0) = 3$ $\quad \left[x = e^{2t} + e^{3t} + t^2 + t\right]$
   $y' = 2y - 2t - 1$ $\quad \left[y = 2e^{2t} + t + 1\right]$

2 Seminář - integrální počet

2.1 Funkční řady

2.1.1 Najděte obor konvergence řady $\sum\limits_{n=1}^{\infty} u_n(x)$, je-li

1. $u_n(x) = \ln^n x$ $\quad \left[\frac{1}{e} < x < e\right]$
2. $u_n(x) = \frac{(-1)^n}{2n+1}(\frac{1-x}{1+x})^n$ $\quad [< 0, \infty)]$
3. $u_n(x) = \frac{1}{1+x^n}$ $\quad [x \in R- < -1, 1 >]$
4. $u_n(x) = \frac{x^n}{1+x^{2n}}$ $\quad [x \in R - \{-1, 1\}]$
5. $u_n(x) = \frac{(-1)^{n+1}}{x^n}$ $\quad [|x| > 1]$
6. $u_n(x) = e^{-nx}$ $\quad [x > 0]$
7. $u_n(x) = \frac{\cos nx}{e^{nx}}$ $\quad [x > 0]$
8. $u_n(x) = (5 - x^2)^n$ $\quad \left[2 < |x| < \sqrt{6}\right]$
9. $u_n(x) = n^{-\ln x^2}$ $\quad [|x| > \sqrt{e}]$
10. $u_n(x) = n^2 e^{-nx^2}$ $\quad [x \in R - \{0\}]$
11. $u_n(x) = \frac{x^n}{1-x^n}$ $\quad [|x| < 1]$

2.1.2 Rozhodněte o stejnoměrné konvergenci posloupnosti $\{f_n(x)\}$, je-li

1. $f_n(x) = \frac{n^2}{n^2+x^2}$ $\quad [x \in R$ nestejnoměrně, $x \in< -1, 1 >$ stejnoměrně$]$
2. $f_n(x) = x^n$ $\quad \left[x \in< 0, 1 >\right.$ nestejnoměrně, $x \in< 0, \frac{1}{2} >$ stejnoměrně$\left.\right]$
3. $f_n(x) = \frac{\operatorname{arctg} nx}{\sqrt{n+x}}$ $\quad [x \in< 0, +\infty)$ stejnoměrně$]$
4. $f_n(x) = x^n - x^{n+1}$ $\quad [x \in< 0, 1 >$ stejnoměrně$]$
5. $f_n(x) = x^n - x^{2n}$ $\quad [x \in< 0, 1 >$ nestejnoměrně$]$
6. $f_n(x) = \frac{nx}{1+n^2x^2}$ $\quad [x \in< 0, 2 >$ nestejnoměrně$]$
7. $f_n(x) = \frac{2x}{1+n^2x^2}$ $\quad [x \in R$ stejnoměrně$]$

8. $f_n(x) = \sqrt{x + \frac{1}{n}} - \sqrt{x}$ $[x \in < 0, +\infty >$ stejnoměrně$]$

9. $f_n(x) = e^{n(x-1)}$ $[x \in (0,1)$ nestejnoměrně$]$

10. $f_n(x) = \operatorname{arctg} nx$ $[x \in (0,+\infty)$ nestejnoměrně$]$

11. $f_n(x) = x \operatorname{arctg} nx$ $[x \in (0,+\infty)$ stejnoměrně$]$

2.1.3 Dokažte stejnoměrnou konvergenci $\sum\limits_{n=1}^{\infty} u_n(x)$, je-li

1. $u_n(x) = \frac{1}{x^2+n^2}$ $[x \in R]$

2. $u_n(x) = \frac{(-1)^n}{x+2^n}$ $[x \geq 0]$

3. $u_n(x) = \frac{x}{1+n^4x^2}$ $[x \in R]$

4. $u_n(x) = \frac{\sin nx}{\sqrt[3]{n^4+x^4}}$ $[x \in R]$

5. $u_n(x) = \frac{nx}{1+n^5x^2}$ $[x \in R]$

6. $u_n(x) = \operatorname{arctg} \frac{2x}{x^2+n^3}$ $[x \in R]$

7. $u_n(x) = \frac{\cos nx}{n^2}$ $[x \in R]$

8. $u_n(x) = \frac{x^2 \sin(n\sqrt{x})}{1+n^3x^4}$ $[x \geq 0]$

9. $u_n(x) = (\operatorname{arctg} \frac{x}{x^2+n^2})^2$ $[x \geq 0]$

10. $u_n(x) = \ln(1 + \frac{x^2}{n \ln^2 n})$ $[n \geq 2, |x| \leq a, a > 0]$

11. $u_n(x) = \frac{\sin(\frac{x^2}{n})}{x^2\sqrt{n}+1}$ $[|x| \leq a, a > 0]$

12. $u_n(x) = \frac{\sin(\frac{x}{n}) \sin 2nx}{x^2+4n}$ $[x \in R]$

13. $u_n(x) = \frac{n^2}{\sqrt{n!}}(x^n + x^{-n})$ $\left[\frac{1}{2} \leq |x| \leq 2\right]$

14. $u_n(x) = x^2 e^{-nx}$ $[\varepsilon \leq x \leq a, (\varepsilon, a > 0, \varepsilon < a)]$

2.2 Mocniné řady

2.2.1 Najděte poloměr konvergence řady $\sum\limits_{n=0}^{\infty} a_n x^n$, je-li

1. $a_n = \frac{1}{n^2}$ $[1]$

2. $a_n = \frac{1}{n!}$ $[\infty]$

3. $a_n = \frac{(1+i)^n}{n2^n}$ $[\sqrt{2}]$

4. $a_n = \alpha^{n^2} (0 < \alpha < 1)$ [∞]

5. $a_n = \frac{a^n}{n} + \frac{b^n}{n^2} (a, b > 0)$ [$\min(\frac{1}{a}, \frac{1}{b})$]

6. $a_n = \frac{3^{-\sqrt{n}}}{\sqrt{n^2+1}}$ [1]

7. $a_n = \frac{1}{a^n+b^n} (a, b > 0)$ [$\min(a, b)$]

8. $a_n = (-1)^{n-1} \{\frac{2^n (n!)^2}{(2n+1)!}\}^p$ [2^p]

9. $a_n = \frac{(-1)^{n-1}}{n!} (\frac{n}{e})^n$ [1]

10. $a_n = \frac{a(a+1)\dots(a+n-1)b(b+1)\dots(b+n-1)}{n!c(c+1)\dots(c+n-1)}$ [1]

2.2.2 Najděte poloměr konvergence řady

1. $\sum_{n=1}^{\infty} \frac{[3+(-1)^n]^n}{n} x^n$ [$\frac{1}{4}$]

2. $\sum_{n=1}^{\infty} 5^n x^{3n}$ [$\frac{1}{\sqrt[3]{5}}$]

3. $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n} x^{2n}$ [$\sqrt{\frac{e}{2}}$]

4. $\sum_{n=1}^{\infty} 3^n (n^3 + 2) x^{2n}$ [$\frac{1}{\sqrt{3}}$]

2.2.3 Najděte obor konvergence mocninné řady $\sum_{n=0}^{\infty} a_n (x - x_0)^n$, je-li

1. $a_n = \frac{1}{n\sqrt{n}}, x_0 = 1$ [$< 0, 2 >$]

2. $a_n = (\frac{2n-1}{3n+2})^n, x_0 = -2$ [$(-\frac{7}{2}, -\frac{1}{2})$]

3. 6.cv $a_n = \frac{(-1)^n}{2n+1}, x_0 = 0$ [$(-1, 1 >$]

4. $a_n = \frac{1}{\sqrt[3]{n} 3^n}, x_0 = 1$ [$< -2, 4)$]

5. $a_n = \sqrt{\frac{n^4+3}{n^3+4n}}, x_0 = -2$ [$(-3, -1)$]

6. $a_n = \frac{5^n+(-3)^n}{n+1}, x_0 = 0$ [$< -\frac{1}{5}, \frac{1}{5})$]

7. $a_n = \frac{1}{\sqrt{n+1}} \ln \frac{3n-2}{3n+2}, x_0 = -1$ [$< -2, 0 >$]

8. $a_n = \frac{\sqrt[3]{2n+1} - \sqrt[3]{2n-1}}{\sqrt{n}}, x_0 = -3$ [$< -4, -2 >$]

9. $a_n = \sqrt[n]{a} - 1, x_0 = 0, a > 0, a \neq 1$ [$< -1, 1)$]

10. $a_n = \frac{3^{-\sqrt{n}}}{\sqrt{n^2+n+1}}, x_0 = 1$ [$< 0, 2 >$]

2.2.4 Najděte rozvoj funkce $f(x)$ v mocninou řadu

1. $f(x) = e^{-x^2}$ $\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}; x \in R\right]$

2. $f(x) = \cos^2 x$ $\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n-1}}{(2n)!} x^{2n}; x \in R\right]$

3. $f(x) = \sin 3x \sin 5x$ $\left[\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!}(1 - 2^{4n}) x^{2n}; x \in R\right]$

4. $f(x) = \sin^3 x$ $\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3(3^{2n}-1)}{4(2n+1)!} x^{2n+1}; x \in R\right]$

5. $f(x) = \frac{x^2}{(1+x)^2}$ $\left[\sum_{n=0}^{\infty} (-1)^n (n+1) x^{n+2}; x \in (-1,1)\right]$

6. $f(x) = \frac{5x-4}{x+2}$ $\left[-2 + \sum_{n=1}^{\infty} \frac{7(-1)^{n-1}}{2^n} x^n; x \in (-2,2)\right]$

7. $f(x) = \frac{1}{x^2-2x-3}$ $\left[-\frac{1}{4} \sum_{n=0}^{\infty} \frac{1+(-1)^n 3^{n+1}}{3^{n+1}} x^n; x \in (-1,1)\right]$

8. $f(x) = \ln \sqrt{\frac{1+x}{1-x}}$ $\left[\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}; x \in (-1,1)\right]$

9. $f(x) = \ln \frac{3-2x}{2+3x}$ $\left[\ln \frac{3}{2} + \sum_{n=1}^{\infty} \{(-\frac{3}{2})^n - (\frac{2}{3})^n\} \frac{x^n}{n}; x \in (-\frac{2}{3}, \frac{2}{3} >\right]$

10. $f(x) = \frac{1}{\sqrt{1-x^2}}$ $\left[1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^{2n}; x \in (-1,1)\right]$

11. $f(x) = \sqrt{1+x^2}$ $\left[1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-3)!!}{(2n)!!} x^{2n}; x \in (-1,1)\right]$

12. $f(x) = (1-x^2)^{-\frac{3}{2}}$ $\left[\sum_{n=0}^{\infty} \frac{(2n+1)!!}{(2n)!!} x^{2n}; x \in (-1,1)\right]$

13. $f(x) = \frac{x}{\sqrt{1-2x}}$ $\left[x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{n!} x^{n+1}; x \in (-\frac{1}{2}, \frac{1}{2})\right]$

14. $f(x) = (1+x^2) \operatorname{arctg} x$ $\left[x + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} x^{2n+1}; x \in < -1, 1 >\right]$

2.2.5 Najděte rozvoj $f(x)$ v mocninnou řadu

1. $f(x) = \ln(x + \sqrt{1+x^2})$ $\left[x + \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!} \frac{x^{2n+1}}{2n+1}; x \in < -1, 1 >\right]$

2. $f(x) = \arcsin x$ $\left[x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{x^{2n+1}}{2n+1}; x \in < -1, 1 >\right]$

3. $f(x) = \operatorname{arctg} \frac{x+3}{x-3}$ $\left[-\frac{\pi}{4} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{3^{2n+1}} \frac{x^{2n+1}}{2n+1}; x \in < -3, -3 > \right]$

4. $f(x) = \frac{1}{1-x-x^2}$ $\left[\sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \{ (\frac{\sqrt{5}+1}{2})^{n+1} + (-1)^n (\frac{\sqrt{5}-1}{2})^{n+1} \}; |x| < \frac{\sqrt{5}-1}{2} \right]$

5. $f(x) = \frac{1}{1+x+x^2}$ $\left[\frac{2}{\sqrt{3}} \sum_{n=0}^{\infty} \sin \frac{2\pi(n+1)}{3} x^n; x \in (-1, 1) \right]$

6. $f(x) = \frac{x \cos \alpha - x^2}{1 - 2x \cos \alpha + x^2}$ $\left[\sum_{n=1}^{\infty} x^n \cos n\alpha; x \in (-1, 1) \right]$

7. $f(x) = \frac{1}{4} \ln \frac{1+x}{1-x} + \frac{1}{2} \operatorname{arctg} x$ $\left[\sum_{n=0}^{\infty} \frac{x^{4n+1}}{4n+1}; x \in (-1, 1) \right]$

8. $f(x) = x \operatorname{arctg} x - \ln \sqrt{1+x^2}$ $\left[\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{2n(2n-1)}; x \in < -1, 1 > \right]$

9. $f(x) = x \arcsin x + \sqrt{1-x^2}$ $\left[1 + \frac{x^2}{2} + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n+2)!!} \frac{x^{2n+2}}{2n+1}; x \in < -1, 1 > \right]$

10. $f(x) = \frac{\ln(1+x)}{1+x}$ $\left[\sum_{n=1}^{\infty} (-1)^{n-1} (1 + \frac{1}{2} + \ldots + \frac{1}{n}) x^n; x \in (-1, 1) \right]$

11. $f(x) = \frac{e^x}{1-x}$ $\left[\sum_{n=0}^{\infty} \sum_{k=0}^{u} \frac{1}{k!} x^n; x \in (-1, 1) \right]$

12. $f(x) = \operatorname{arctg}^2 x$ $\left[\sum_{n=1}^{\infty} (-1)^{n-1} (1 + \frac{1}{3} + \ldots + \frac{1}{2n-1}) \frac{x^{2n}}{n}; x \in < -1, 1 > \right]$

13. $f(x) = e^x \sin x$ $\left[\sum_{n=1}^{\infty} \frac{2^{\frac{n}{2}} \sin(\frac{n\pi}{4})}{n!} x^n; x \in R \right]$

14. $f(x) = e^x \cos x$ $\left[\sum_{n=1}^{\infty} \frac{2^{\frac{n}{2}} \cos(\frac{n\pi}{4})}{n!} x^n; x \in R \right]$

15. $f(x) = (\frac{\arcsin x}{x})^2$ $\left[\sum_{n=0}^{\infty} \frac{2^{2n+1}(n!)^2}{(2n+2)!} x^{2n}; |x| \le 1 \right]$

2.2.6 Vypočtěte integrály

1. $\int_0^x e^{-t^2} dt$ $\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}; x \in R \right]$

2. $\int_0^x \frac{\sin t}{t} dt$ $\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}; x \in R \right]$

3. $\int_0^x \frac{dt}{\sqrt{1-t^4}}$ $\left[x + \sum_{n=1}^{\infty} \frac{(2n-1)!! x^{4n+1}}{(2n)!!(4n+1)}; x \in (-1, 1) \right]$

4. $\int_0^x \frac{t^2 dt}{\sqrt{1+t^2}}$ $\left[\frac{x^3}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!! x^{2n+3}}{(2n)!!(2n+3)}; x \in < -1, 1 > \right]$

2.3 Fourierovy řady

2.3.1 Najděte Fourierovu řadu funkcí $f_n(x) = \sin^n x$ a $g_n(x) = \cos^n x$ pro $n = 2, 3, 4, 5.$

- $f_2(x) = \frac{1}{2} - \frac{1}{2}\cos 2x$ $\qquad \left[g_2(x) = \frac{1}{2} + \frac{1}{2}\cos 2x\right]$
- $f_3(x) = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$ $\qquad \left[g_3(x) = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x\right]$
- $f_4(x) = \frac{3}{4} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$ $\qquad \left[g_4(x) = \frac{3}{4} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right]$
- $f_5(x) = -\frac{5}{8}\sin x + \frac{5}{16}\sin 3x - \frac{1}{16}\sin 5x$ $\qquad \left[g_5(x) = \frac{5}{8}\cos x + \frac{5}{16}\cos 3x + \frac{1}{16}\cos 5x\right]$

2.3.2 Najděte Fourierovu řadu funkce $f(x)$ na intervalu $(-\pi, \pi)$, je-li

1. $f(x) = x$ $\qquad \left[2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}\right]$
2. $f(x) = 1$ pro $0 \le x \le \pi$ $\qquad \left[\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}\right]$

   $f(x) = 0$ pro $-\pi \le x \le 0$
3. $f(x) = |x|$ Výsledku využijte k sečtení řady $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$ $\qquad \left[\frac{\pi}{2} - 4\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}; \frac{\pi^2}{8}\right]$
4. $f(x) = \pi^2 - x^2$ Výsledku využijte k sečtení řady $\sum_{n=1}^{\infty}\frac{1}{n^2}, \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$

   $\left[\frac{2}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx; \frac{\pi^2}{6}, \frac{\pi^2}{12}\right]$
5. $f(x) = \operatorname{sign} x$ Výsledku využijte k sečtení řady $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ $\qquad \left[\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}; \frac{\pi}{4}\right]$
6. $f(x) = \sin ax \quad a \notin Z$ $\qquad \left[\frac{2\sin\pi a}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n\sin nx}{n^2-a^2}\right]$
7. $f(x) = \cos ax \quad a \notin Z$ $\qquad \left[\frac{2\sin\pi a}{\pi}\{\frac{1}{2a} + \sum_{n=1}^{\infty}(-1)^n\frac{a\cos nx}{a^2-n^2}\}\right]$
8. $f(x) = e^{ax} \quad a \neq 0$ $\qquad \left[\frac{2}{\pi}\sinh a\pi\{\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx)\}\right]$
9. $f(x) = \frac{q\sin x}{1-2q\cos x+q^2} \quad |q| < 1$ $\qquad \left[\sum_{n=1}^{\infty}q^n\sin nx; \text{zaveďte } e^{ix} = z\right]$

2.3.3 Najděte Fourierovu řadu funkce $f(x)$, je-li

1. $f(x) = \frac{\pi-x}{2}, x \in (0, 2\pi)$ $\qquad \left[\sum_{n=1}^{\infty}\frac{\sin nx}{n}\right]$
2. $f(x) = x, x \in (a, a+2l)$ $\qquad \left[a + l + \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{\pi}(\sin\frac{n\pi a}{l}\cos\frac{n\pi x}{l} - \cos\frac{n\pi a}{l}\sin\frac{n\pi x}{l})\right]$

3. $f(x) = x^2, x \in (0, 2\pi)$ $\left[\frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - 4\pi \sum_{n=1}^{\infty} \frac{\sin nx}{n}\right]$

4. $f(x) = e^{ax}, x \in (-h, h)$ $\left[2\sinh ah\{\frac{1}{2ah} + \sum_{n=1}^{\infty} (-1)^n \frac{ah\cos(\frac{n\pi x}{h}) - n\sin(\frac{n\pi x}{h})}{(ah)^2 + (\pi n)^2}\}\right]$

5. $f(x) = x\cos x, x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $\left[\frac{16}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{(4n^2-1)^2} \sin 2nx\right]$

6. $f(x) = e^x - 1, x \in (0, 2\pi)$ $\left[\frac{e^{2\pi}-1}{\pi}\{\frac{1}{2} + \sum_{n=1}^{\infty} (\frac{\cos nx}{1+n^2} - \frac{n\sin nx}{1+n^2})\} - 1\right]$

2.3.4 Najděte Fourierovu řadu funkce $f(x)$, je-li

1. $f(x) = \frac{\pi}{4} - \frac{x}{2}, x \in (0, \pi)$ (kosinová řada) $\left[\frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}\right]$

2. $f(x) = x^2, x \in (0, \pi)$ (sinová řada) $\left[\frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1}\{\frac{\pi^2}{n} + \frac{2}{n^2}[(-1)^n - 1]\} \sin nx\right]$

3. $f(x) = \sin ax, a \in Z, x \in (0, \pi)$ (kosinová řada) $\left[\frac{4a}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{a^2-(2n+1)^2} \text{pro } a \text{ sudé}\right]$

$\left[\frac{4a}{\pi}\{\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{\cos 2nx}{a^2-4n^2}\} \text{pro } a \text{ liché}\right]$

4. $f(x) = \cos ax, a \in Z, x \in (0, \pi)$ (sinová řada) $\left[-\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{a^2-(2n+1)^2} \text{ pro } a \text{ sudé}\right]$

$\left[-\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n\sin 2nx}{a^2-4n^2} \text{ pro } a \text{ liché}\right]$

5. $f(x) = x(\frac{\pi}{2} - x), x \in (0, \frac{\pi}{2})$ podle soustavy

$\{\cos(2n-1)x\}, n \in N$ $\left[-2\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\{1 + \frac{4(-1)^n}{(2n-1)\pi}\} \cos(2n-1)x\right]$

$\{\sin(2n-1)x\}, n \in N$ $\left[\sum_{n=1}^{\infty}\{\frac{2(-1)^n}{(2n-1)^2} + \frac{8}{(2n-1)^3}\} \sin(2n-1)x\right]$

2.3.5 Integrací Fourierova rozvoje funkce $f(x) = x$ najděte rozvoj funkcí x^2, x^3, x^4, x^5 pro $x \in (-\pi, \pi)$

1. $f(x) = x$ $\left[2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}\right]$

2. $f(x) = x^2$ $\left[\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} (-1)^n\right]$

3. $f(x) = x^3$ $\left[2\sum_{n=1}^{\infty} (-1)^n \frac{6-\pi^2 n^2}{n^3} \sin nx\right]$

4. $f(x) = x^4$ $\left[\frac{\pi^4}{5} + 8\sum_{n=1}^{\infty}(-1)^{n+1}\frac{6-\pi^2 n^2}{n^4}\cos nx\right]$

5. $f(x) = x^5$ $\left[2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{120-20\pi^2 n^2+\pi^4 n^4}{n^5}\sin nx\right]$