Dvojné a trojné integrály.

1. Najděte meze v integrálu $\iint_D f(x, y)\,dx\,dy$, je-li D omezena

a) $x = 0$, $x = 3$, $y = -2$, $y = 2$ / $0 \leqq x \leqq 3$, $-2 \leqq y \leqq 2$ /

b) $x = 3$, $x = 5$, $3x - 2y + 4 = 0$, $3x - 2y + 1 = 0$

/ $3 \leqq x \leqq 5$, $\frac{3x+1}{2} \leqq y \leqq \frac{3x+4}{2}$ /

c) $x = 0$, $y = 0$, $x + y = 2$ / $0 \leqq x \leqq 2$, $0 \leqq y \leqq 2 - x$ /

d) D je trojúhelník s vrcholy $0[0, 0]$, $A[2, 1]$, $B[-2, 1]$

/ $0 \leqq y \leqq 1$, $-2y \leqq x \leqq 2y$ /

e) D je čtyřúhelník s vrcholy $0[0, 0]$, $A[1, 0]$, $B[1, 2]$, $C[0, 1]$

/ $0 \leqq x \leqq 1$, $0 \leqq y \leqq x + 1$ /

f) $x^2 + y^2 \leqq 1$, $x \geqq 0$, $y \geqq 0$ / $0 \leqq x \leqq 1$, $0 \leqq y \leqq \sqrt{1 - x^2}$ /

g) $(x - 2)^2 + (y - 3)^2 \leqq 4$ / $0 \leqq x \leqq 4$, $3 - \sqrt{4x - x^2} \leqq y \leqq 3 + \sqrt{4x - x^2}$ /

h) $y - 2x \leqq 0$, $2y - x \geqq 0$, $xy \leqq 2$ / $0 \leqq x \leqq 1$, $\frac{x}{2} \leqq y \leqq 2x$; $1 \leqq x \leqq 2$,

$\frac{x}{2} \leqq y \leqq \frac{2}{x}$ /

2. Zaměňte pořadí integrace v integrálech

a) $\int_0^1 dx \int_{x^3}^{x^2} f(x, y)\,dy$ / $\int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} f(x, y)\,dx$

b) $\int_{-1}^1 dx \int_0^{\sqrt{1-x^2}} f(x, y)\,dy$ / $\int_0^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y)\,dx$ /

c) $\int_1^2 dx \int_{2-x}^{\sqrt{2x-x^2}} f(x, y)\,dy$ / $\int_0^1 dy \int_{2-y}^{1+\sqrt{1-y^2}} f(x, y)\,dx$ /

d) $\int_{-2}^2 dx \int_{-\frac{1}{\sqrt{2}}\sqrt{4-x^2}}^{\frac{1}{\sqrt{2}}\sqrt{4-x^2}} f(x, y)\,dy$ / $\int_{-\sqrt{2}}^{\sqrt{2}} dy \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} f(x, y)\,dx$ /

e) $\int_0^r dx \int_x^{\sqrt{2rx-x^2}} f(x, y)\,dy$ / $\int_0^r dy \int_{r-\sqrt{r^2-y^2}}^{y} f(x, y)\,dx$ /

f) $\int_0^2 dx \int_x^{2x} f(x, y)\, dy$ / $\int_0^2 dy \int_{\frac{y}{2}}^{y} f(x, y)\, dx +$

$+ \int_2^4 dy \int_{\frac{y}{2}}^{2} f(x, y)\, dx$ /

g) $\int_0^2 dx \int_{2x}^{6-x} f(x, y)\, dy$ / $\int_0^4 dy \int_0^{y/2} f(x, y)\, dx +$

$+ \int_4^6 dy \int_{y/2}^{2} f(x, y)\, dx$ /

h) $\int_{-6}^{2} dx \int_{\frac{x^2}{4}-1}^{2-x} f(x, y)\, dy$ / $\int_{-1}^{0} dy \int_{-2\sqrt{1+y}}^{2\sqrt{1+y}} f(x, y)\, dx +$

$+ \int_0^8 dy \int_{-2\sqrt{1+y}}^{2-y} f(x, y)\, dx$ /

i) $\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{1-x^2} f(x, y)\, dy$ / $\int_{-1}^{0} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y)\, dx +$

$+ \int_0^1 dy \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x, y)\, dx$ /

j) $\int_0^1 dx \int_0^{x^{2/3}} f(x, y)\, dy + \int_1^2 dx \int_0^{1-\sqrt{4x-x^2-3}} f(x, y)\, dy$

/ $\int_0^1 dy \int_{y^{3/2}}^{2-\sqrt{2y-y^2}} f(x, y)\, dx$ /

k) $\int_0^1 dx \int_0^{1-x} dy \int_0^{x+y} f(x,y,z)\, dz$ / Na příklad

$$\int_0^1 dz \left\{ \int_0^2 dy \int_{z-y}^{1-y} f\, dx + \int_z^1 dy \int_0^{1-y} f\, dx \right\}$$ /

l) $\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\, dz$ / Na příklad

$$\int_0^1 dz \int_{-z}^{z} dy \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x,y,z)\, dx$$ /

m) $\int_0^1 dx \int_0^1 dy \int_0^{x^2+y^2} f(x,y,z)\, dz$ / Na příklad

$$\int_0^1 dz \left\{ \int_0^{\sqrt{z}} dy \int_{\sqrt{z-y^2}}^{1} f\, dx + \int_{\sqrt{z}}^{1} dy \int_0^1 f\, dx \right\} +$$

$$+ \int_1^2 dz \int_{\sqrt{z-1}}^{1} dy \int_{\sqrt{z-y^2}}^{1} f\, dx$$ /

3. Vypočtěte integrály

a) $\iint_D xy\, dx\, dy$, $D: 0 \leqq x \leqq 1$, $0 \leqq y \leqq 2$ / 1 /

b) $\iint_D \frac{x^2}{1+y^2}\, dx\, dy$, $D: 0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ / $\frac{\pi}{12}$ /

c) $\iint_D \frac{y\, dx\, dy}{(1+x^2+y^2)^{3/2}}$, $D: 0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ / $\ln \frac{2+\sqrt{2}}{1+\sqrt{3}}$ /

d) $\iint\limits_D x \sin(x + y)\, dx\, dy$, D: $0 \leqq x \leqq \pi$, $0 \leqq y \leqq \frac{\pi}{2}$ / $\pi - 2$ /

e) $\iint\limits_D x^2\, y\, e^{xy}\, dx\, dy$, D: $0 \leqq x \leqq 1$, $0 \leqq y \leqq 2$ / 2 /

f) $\iint\limits_D dx\, dy$, D: $x = a$, $y^2 = x$, $y = 0$ / $\frac{2}{3} a^{3/2}$ /

g) $\iint\limits_D x^3\, y^2\, dx\, dy$, D: $x^2 + y^2 \leqq R^2$ / 0 /

h) $\iint\limits_D xy^2\, dx\, dy$, D: $y^2 = 2px$, $x = \frac{p}{2}$ $(p > 0)$ / $\frac{p^5}{21}$ /

i) $\iint\limits_D \frac{x^2}{y^2}\, dx\, dy$, D: $x = 2$, $y = x$, $xy = 1$ / $\frac{9}{4}$ /

j) $\iint\limits_D \cos(x + y)\, dx\, dy$, D: $x = 0$, $y = \pi$, $y = x$ / -2 /

k) $\iint\limits_D \sqrt{1 - x^2 - y^2}\, dx\, dy$, D: $x^2 + y^2 \leqq 1$, $x \geqq 0$, $y \geqq 0$ / $\frac{\pi}{6}$ /

l) $\iint\limits_D (x^2 + y^2)\, dx\, dy$, D: $y = x$, $y = x + a$, $y = a$, $y = 3a$
$(a > 0)$ / $14\, a^4$ /

m) $\iint\limits_D (\,|x| + |y|\,)\, dx\, dy$, D: $|x| + |y| \leqq 1$ / $\frac{4}{3}$ /

n) $\iint\limits_D (x + y)\, dx\, dy$, D: $y^2 = 2x$, $x + y = 4$, $x + y = 12$
/ $543 + \frac{11}{25}$ /

o) $\iint_D xy \, dx \, dy$, D: $xy = 1$, $x + y = \frac{5}{2}$ / $\frac{165}{128} - \ln 2$ /

p) $\iint_D x^2 y^2 \sqrt{1 - x^3 - y^3} \, dx \, dy$, D: $x^3 + y^3 \leqq 1$, $x \geqq 0$, $y \geqq 0$,
/ $\frac{4}{135}$ /

4. Vypočtěte integrály

a) $\iiint_V dx \, dy \, dz$, V: $0 \leqq x \leqq 1$, $0 \leqq y \leqq 2$, $0 \leqq z \leqq 3$ / 6 /

b) $\iiint_V xyz \, dx \, dy \, dz$, V: $z = y$, $y = x$, $x = 0$, $y = 0$, $z = 0$,
$x = a$ / $\frac{a^6}{48}$ /

c) $\iiint_V \frac{dx \, dy \, dz}{(1 + x + y + z)^3}$, V: $x + y + z = 1$, $x = 0$, $y = 0$, $z = 0$
/ $\frac{1}{2} \ln 2 - \frac{5}{16}$ /

d) $\iiint_V xyz \, dx \, dy \, dz$, V: $x^2 + y^2 + z^2 = 1$, $x \geqq 0$, $y \geqq 0$, $z \geqq 0$
/ $\frac{1}{48}$ /

e) $\iiint_V \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dx \, dy \, dz$, V: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqq 1$ / $\frac{4}{5}\pi abc$ /

f) $\iiint_V xy \, dx \, dy \, dz$, V: $z = xy$, $x + y = 1$, $z \geqq 0$ / $\frac{1}{180}$ /

5. Zavedením polárních souřadnic transformujte obor D .

a) D: $x^2 + y^2 \leqq R^2$ / $0 \leqq r \leqq R$, $0 \leqq \varphi \leqq 2\pi$ /

b) D: $x^2 + y^2 \leqq ax$ / $-\frac{\pi}{2} \leqq \varphi \leqq \frac{\pi}{2}$, $0 \leqq r \leqq a \cos \varphi$ /

c) D: $x^2 + y^2 \leqq by$ / $0 \leqq \varphi \leqq \pi$, $0 \leqq r \leqq b \sin \varphi$ /

d) D: $x^2 + y^2 = 4x$, $x^2 + y^2 = 8x$, $y = x$, $y = 2x$
/ $\frac{\pi}{4} \leqq \varphi \leqq \operatorname{arctg} 2$, $4 \cos \varphi \leqq r \leqq 8 \cos \varphi$ /

e) D: $y = x$, $y = 0$, $x = 1$ / $0 \leqq \varphi \leqq \frac{\pi}{4}$, $0 \leqq r \leqq \frac{1}{\cos\varphi}$ /

f) D: $x^2 + y^2 \leqq 4$, $x + y \geqq 2$ / $0 \leqq \varphi \leqq \frac{\pi}{2}$, $\frac{\sqrt{2}}{\sin(\varphi + \frac{\pi}{4})} \leqq r \leqq 2$ /

. Zaveďte polární souřadnice do integrálu

a) $\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x,y)\, dy$ / $\int_0^{\pi/2} d\varphi \int_0^R f(r\cos\varphi, r\sin\varphi) r\, dr$ /

b) $\iint_D f(\sqrt{x^2 + y^2})\, dx\, dy$, D: $x^2 + y^2 \leqq 1$ / $2\pi \int_0^1 r f(r)\, dr$ /

c) $\int_{R/2}^{2R} dy \int_0^{\sqrt{2Ry-y^2}} f(x,y)\, dx$ / $\int_{\pi/6}^{\pi/2} d\varphi \int_{\frac{R}{2\sin\varphi}}^{2R\sin\varphi} r f(r\cos\varphi, r\sin\varphi)\, dr$ /

d) $\int_0^{\frac{R}{\sqrt{1+R^2}}} dx \int_0^{Rx} f(\frac{y}{x})\, dy + \int_{\frac{R}{\sqrt{1+R^2}}}^{R} dx \int_0^{\sqrt{R^2-x^2}} f(\frac{y}{x})\, dy$

/ $\frac{R^2}{2} \int_0^{\mathrm{arctg}\, R} f(\mathrm{tg}\,\varphi)\, d\varphi$ /

e) $\int_0^1 dx \int_0^1 f(x,y)\, dy$ / $\int_0^{\pi/4} d\varphi \int_0^{1/\cos\varphi} r f(r\cos\varphi, r\sin\varphi)\, dr$ ·

$+ \int_{\pi/4}^{\pi/2} d\varphi \int_0^{1/\sin\varphi} r f(r\cos\varphi, r\sin\varphi)\, dr$ /

f) $\int_0^1 dx \int_{1-x}^{\sqrt{1-x^2}} f(x,y)\, dy$ / $\int_0^{\pi/2} d\varphi \int_{1/\sqrt{2}\sin(\varphi+\pi/4)}^{1} r f(r\cos\varphi, r\sin\varphi)\, dr$ /

7. Do integrálu $\iiint\limits_V f(x, y, z)\, dx\, dy\, dz$ zaveďte cylindrické nebo sférické souřadnice.

a) V: $x^2 + y^2 = R^2$, $z = 0$, $z = 1$, $y = x$, $y = x\sqrt{3}$ (1. oktant)

/ $\int_0^1 dz \int_{\pi/4}^{\pi/3} d\varphi \int_0^R r\, f(r\cos\varphi,\, r\sin\varphi,\, z)\, dr$ /

b) V: $x^2 + y^2 = 2x$, $z = 0$, $z = x^2 + y^2$

/ $\int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2\cos\varphi} r\, dr \int_0^{r^2} f(r\cos\varphi,\, r\sin\varphi,\, z)\, dz$ /

c) V: $x^2 + y^2 + z^2 \leqq R^2$ v 1. oktantu

/ $\int_0^{\pi/2} \cos\vartheta\, d\vartheta \int_0^{\pi/2} d\varphi \int_0^R f(r\cos\varphi\cos\vartheta,\, r\sin\varphi\cos\vartheta,\, r\sin\vartheta) r^2\, dr$ /

d) V: $x^2 + y^2 + z^2 \leqq R^2$, $x^2 + y^2 + (Z - R)^2 \leqq R^2$

/ $\int_0^{2\pi} d\varphi \left\{ \int_0^{\pi/6} \cos\vartheta\, d\vartheta \int_0^{2R\sin\vartheta} r^2 f(r\cos\varphi\cos\vartheta,\, r\sin\varphi\cos\vartheta,\, r\sin\vartheta)\, dr \right.$

$\left. + \int_{\pi/6}^{\pi/2} \cos\vartheta\, d\vartheta \int_0^R r^2 f(r\cos\varphi\cos\vartheta,\, r\sin\varphi\cos\vartheta,\, r\sin\vartheta)\, dr \right\}$ /

8. Užitím věty o substituci vypočtěte

a) $\iint\limits_D \sqrt{x^2 + y^2}\, dx\, dy$, D: $x^2 + y^2 \leqq a^2$ / $\frac{2\pi a^3}{3}$ /

b) $\iint\limits_D \sin\sqrt{x^2 + y^2}\, dx\, dy$, D: $\pi^2 \leqq x^2 + y^2 \leqq 4\pi^2$ / $-6\pi^2$ /

c) $\int_0^R dx \int_0^{\sqrt{R^2 - x^2}} \ln(1 + x^2 + y^2)\, dx\, dy$ / $\frac{\pi}{4}\left\{(1 + R^2)\ln(1 + R^2) - R^2\right\}$ /

d) $\iint\limits_D \sqrt{\frac{1 - x^2 - y^2}{1 + x^2 + y^2}}\, dx\, dy$, D: $x^2 + y^2 \leqq 1$, $x \geqq 0$, $y \geqq 0$ / $\frac{\pi(\pi - 2)}{8}$ /

e) $\iint\limits_D \sqrt{R^2 - x^2 - y^2}\, dx\, dy$, D: $x^2 + y^2 \leqq Rx$ / $\frac{R^3}{3}(\pi - \frac{4}{3})$ /

f) $\iint\limits_D \operatorname{arctg} \frac{y}{x}\, dx\, dy$, D: $x^2 + y^2 \geqq 1$, $x^2 + y^2 \leqq 9$, $y \geqq \frac{x}{\sqrt{3}}$, $y \leqq x\sqrt{3}$ / $\pi^2/6$ /

g) $\iint\limits_D (x + y)\, dx\, dy$, D: $x^2 + y^2 \leqq x + y$ / $\frac{\pi}{2}$ /

h) $\iint\limits_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dx\, dy$, D: $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqq 1$ / $\frac{2}{3}\pi ab$ /

i) $\iint\limits_D (x^2 + y^2)\, dx\, dy$, D: $x^4 + y^4 \leqq 1$ / $\frac{\pi}{\sqrt{2}}$ /

j) $\iint\limits_D \ln \frac{1}{\sqrt{x^2 + y^2}}\, dx\, dy$, D: $x^2 + y^2 \leqq 1$ / $\frac{\pi}{2}$ /

k) $\iint\limits_D \frac{dx\, dy}{\sqrt{x^2 + y^2}}$, D: $x^2 + y^2 \leqq x$ / 2 /

l) $\int_0^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_0^a dz$ / $\frac{\pi a}{2}$ /

m) $\iiint\limits_V z\sqrt{x^2 + y^2}\, dx\, dy\, dz$, V: $x \geqq 0$, $y \geqq 0$, $z = 0$, $z = a$, $x^2 + y^2 = 2x$ / $\frac{8a^2}{9}$ /

n) $\iiint\limits_V \sqrt{x^2 + y^2 + z^2}\, dx\, dy\, dz$, V: $x^2 + y^2 + z^2 \leqq z$ / $\frac{\pi}{10}$ /

o) $\iiint_V (x^2+y^2)\,dx\,dy\,dz$, V: $r^2 \leqq x^2+y^2+z^2 \leqq R^2$, $z \geqq 0$
/ $\frac{4}{15}\pi(R^5-r^5)$ /

p) $\iiint_V \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+(z-2)^2}}$, V: $x^2+y^2+z^2 \leqq 1$ / $\frac{2\pi}{3}$ /

q) $\iiint_V \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+(z-2)^2}}$, V: $x^2+y^2 \leqq 1$, $-1 \leqq z \leqq 1$
/ $\pi\left\{ 3\sqrt{10} + \ln \frac{\sqrt{2}-1}{\sqrt{10}-3} - \sqrt{2} - 8 \right\}$ /

r) $\iiint_V \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\,dx\,dy\,dz$, V: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqq 1$ / $\frac{\pi^2 abc}{4}$ /

s) $\iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz$, V: $x^2+y^2+z^2 \leqq x+y+z$ / $\frac{9\pi}{5}$ /

Užití vícerozměrných integrálů.

1. Vypočtěte obsah rovinného oboru omezeného křivkami

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ / πab /

b) $xy = a^2$, $x + y = \frac{5}{2}a$ $(a > 0)$ / $\left(\frac{15}{8} - 2\ln 2\right) a^2$ /

c) $y^2 = 2px + p^2$, $y^2 = -2qx + q^2$ $(p,q > 0)$ / $\frac{2}{3}(p+q)\sqrt{pq}$ /

d) $(x^2+y^2)^2 = 2a^2(x^2-y^2)$ / $2a^2$ /

e) $(x^2+y^2)^2 = 2ax^3$ / $\frac{5}{8}\pi a^2$ /

f) $(x^2+y^2)^3 = x^4+y^4$ / $\frac{3}{4}\pi$ /

g) $(x^3+y^3)^2 = x^2+y^2$, $x \geqq 0$, $y \geqq 0$ / $\frac{\pi}{6} + \frac{\sqrt{2}}{3}\ln(1+\sqrt{2})$ /

h) $x^3+y^3 = 2xy$, $x \geqq 0$, $y \geqq 0$ / $\frac{2}{3}$ /

i) $(x^2+y^2)^2 = 8a^2xy$, $(x-a)^2+(y-a)^2 \leqq a^2$ $(a > 0)$
/ $a^2\left(\frac{\sqrt{7}}{2} + \arcsin\frac{\sqrt{14}}{8}\right)$ /

j) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{xy}{c^2}$ (a,b,c > 0) / $\frac{a^2b^2}{2c^2}$ /

k) $\left(\frac{x^2}{4}+\frac{y^2}{9}\right)^2=\frac{x^2+y^2}{25}$ / $\frac{39\pi}{25}$ /

2. Vypočtěte objem tělesa omezeného plochami

a) $z=1+x+y$, $z=0$, $x+y=1$, $x=0$, $y=0$ / $\frac{5}{6}$ /

b) $x+y+z=a$, $x^2+y^2=R^2$, $x=0$, $y=0$, $z=0$ $(a \geq R\sqrt{2})$
/ $\frac{\pi R^2 a}{4}-\frac{2}{3}R^3$ /

c) $y=0$, $z=0$, $3x+y=6$, $3x+2y=12$, $x+y+z=6$ / 12 /

d) $x=0$, $y=0$, $z=0$, $x=4$, $y=4$, $z=x^2+y^2+1$ / $186+\frac{2}{3}$ /

e) $z=x^2+y^2$, $z=0$, $y=1$, $y=2x$, $y=6-x$ / $78+\frac{15}{32}$ /

f) $z=9-y^2$, $x=0$, $y=0$, $z=0$, $3x+4y=12$ $(y \geq 0)$ / 45 /

g) $\frac{x^2}{4}+y^2=1$, $z=12-3x-4y$, $z=1$ / 22π /

h) $x^2+y^2=R^2$, $x^2+z^2=R^2$ / $\frac{16}{3}R^3$ /

i) $x^2+y^2=R^2$, $Rz=2R^2+x^2+y^2$, $z=0$ / $\frac{5}{2}\pi R^3$ /

j) $z=x^2-y^2$, $z=0$, $x=3$ / 27 /

3. Vypočtěte objem tělesa omezeného plochami

a) $z=4-y^2$, $z=y^2+2$, $x=-1$, $x=2$ / 8 /

b) $z=x^2+y^2$, $z=x^2+2y^2$, $y=x$, $y=2x$, $x=1$ / $\frac{7}{12}$ /

c) $x^2+y^2+z^2=4$, $x^2+y^2=3z$ / $\frac{19\pi}{6}$ /

d) $x^2+y^2+z^2=R^2$, $x^2+y^2=R(R-2z)$ $(z \geq 0)$ / $\frac{5\pi R^3}{12}$ /

e) $x^2+y^2+z^2=4Rz-3R^2$, $z^2=4(x^2+y^2)$ / $\frac{92\pi R^3}{75}$ /

f) $az=x^2+y^2$, $z=\sqrt{x^2+y^2}$ $(a>0)$ / $\frac{\pi a^3}{6}$ /

g) $x^2+y^2+z^2=2az$, $x^2+y^2 \leq z^2$ / πa^3 /

h) $x^2+y^2+z^2=a^2$, $x^2+y^2+z^2=b^2$, $x^2+y^2=z^2$ $(z \geq 0,\ 0<a<b)$
/ $\frac{\pi}{3}(2-\sqrt{2})(b^3-a^3)$ /

i) $(x^2+y^2+z^2)^3=a^2(x^2+y^2)^2$ / $\frac{64\pi a^3}{105}$ /

j) $(x^2 + y^2 + z^2)^2 = a^3 x$ / $\frac{\pi a^3}{3}$ /

4. Najděte těžiště homogenního oboru

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $y \geq 0$ / 0 ; $\frac{4b}{3\pi}$ /

b) $y = \sin x$, $y = 0$, $x = \frac{\pi}{4}$ / $(1 - \frac{\pi}{4})(\sqrt{2} + 1)$; $\frac{1}{8}(\frac{\pi}{2} - 1)(2 + \sqrt{2})$.

c) $y^2 = x^2 - x^4$ $(x \geq 0)$ / $\frac{3\pi}{16}$; 0 /

d) $x^{2/3} + y^{2/3} = a^{2/3}$ $(x \geq 0$, $y \geq 0)$ / $\frac{256}{315\pi} a$; $\frac{256}{315\pi} a$ /

e) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $y = 0$, $t \in \langle 0 , 2\pi \rangle$ / πa , $\frac{5}{6} a$ /

f) $x = 0$, $y = 0$, $z = 0$, $x = 2$, $y = 4$, $x + y + z = 8$ / $\frac{14}{15}$; $\frac{26}{15}$; $\frac{8}{3}$ /

g) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$ / $\frac{3a}{8}$; $\frac{3b}{8}$; $\frac{3c}{8}$ /

h) $z = \frac{y^2}{2}$, $x = 0$, $y = 0$, $z = 0$, $2x + 3y - 12 = 0$ / $\frac{6}{5}$; $\frac{12}{5}$; $\frac{8}{5}$ /

i) $z = \frac{x^2 + y^2}{2a}$, $x^2 + y^2 + z^2 = 3a^2$ / 0 ; 0 ; $\frac{5a(6\sqrt{3} + 5)}{83}$ /

j) $x^2 + z^2 = a^2$, $y^2 + z^2 = a^2$ $(z \geq 0)$ / 0 ; 0 ; $\frac{3a}{8}$ /

Integrály závislé na parametru.

1. Užitím věty o derivaci podle parametru vypočtěte

a) $$\int_0^\infty \frac{1 - e^{-ax}}{x e^x} \, dx \qquad (a > -1)$$ / $\ln(1 + a)$ /

b) $$\int_0^1 \frac{\operatorname{arctg} ax}{x\sqrt{1 - x^2}} \, dx$$ / $\frac{\pi}{2} \ln(a + \sqrt{1 + a^2})$ /

c) $$\int_0^1 \frac{\ln(1 - a^2 x^2)}{x^2\sqrt{1 - x^2}} \, dx \qquad (|a| < 1)$$ / $\pi(\sqrt{1 - a^2} - 1)$ /

d) $\displaystyle\int_0^\infty \frac{\operatorname{arctg} ax}{x(1+x^2)}\,dx \quad (a \geqq 0)$ $\quad / \frac{\pi}{2}\ln(1+a) \quad /$

e) $\displaystyle\int_0^1 \frac{\ln(1-a^2x^2)}{\sqrt{1-x^2}}\,dx \quad (|a|<1)$ $\quad / \pi \ln \frac{1+\sqrt{1-a^2}}{2} \quad /$

f) $\displaystyle\int_0^{\pi/2} \ln\left(\frac{1+a\sin x}{1-a\sin x}\right)\frac{dx}{\sin x} \quad (|a|<1)$ $\quad / \pi \arcsin a \quad /$

g) $\displaystyle\int_0^\infty \frac{dx}{(x^2+a)^{n+1}} \quad (n\in N,\ a>0)$ $\quad / \frac{\pi}{2}\frac{(2n-1)!!}{(2n)!!}\, a^{-(n+1/2)} \quad /$

h) $\displaystyle\int_0^\infty \frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x}\,dx \quad (\alpha,\beta>0)$ $\quad / \frac{1}{2}\ln\frac{\beta}{\alpha} \quad /$

i) $\displaystyle\int_0^\infty \frac{e^{-\alpha x}-e^{-\beta x}}{x}\sin mx\,dx \quad (\alpha,\beta>0)$ $\quad / \operatorname{arctg}\frac{\beta}{m} - \operatorname{arctg}\frac{\alpha}{m} \quad /$

j) $\displaystyle\int_0^\infty \frac{e^{-\alpha x}-e^{-\beta x}}{x}\cos mx\,dx \quad (\alpha,\beta>0)$ $\quad / \frac{1}{2}\ln\frac{\beta^2+m^2}{\alpha^2+m^2} \quad /$

k) $\displaystyle\int_0^\infty \frac{e^{-ax^2}-e^{-bx^2}}{x^2}\,dx \quad (a,b>0)$ $\quad / \sqrt{\pi}(\sqrt{b}-\sqrt{a}) \quad /$

l) $\displaystyle\int_{-\infty}^\infty e^{-ax^2}\operatorname{ch} bx\,dx \quad (a>0)$ $\quad / \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}} \quad /$

m) $\displaystyle\int_0^\infty e^{-(x^2+\frac{a^2}{x^2})}\,dx \quad (a>0)$ $\quad / \frac{\sqrt{\pi}}{2}\, e^{-2a} \quad /$

2\. Vypočtěte $\int_0^\infty \frac{\sin x}{x}\, dx$ (Užijte vztahu $\frac{\sin x}{x} = \int_0^\infty e^{-xy} \sin x\, dy$)

/ $\frac{\pi}{2}$ /

3\. Ukažte, že Besselova funkce $J_0(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin t)\, dt$ vyhovuje

rovnici $y'' + \frac{y'}{x} + y = 0$

4\. Ukažte, že funkce $u(x, t) = \frac{1}{2}\left\{ f(x - at) + f(x + at)\right\} + \frac{1}{2a} \int_{x-at}^{x+at} F(z)\, dz$

vyhovuje parciální diferenciální rovnici $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

(rovnice kmitání struny) a počátečním podmínkám $u(x, 0) = f(x)$,

$u'_t(x, 0) = F(x)$

5\. Ukažte, že funkce $u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi)\, e^{-\frac{(\xi - x)^2}{4a^2 t}}\, d\xi$

vyhovuje rovnici $\frac{\partial u}{\partial t} = \frac{1}{a^2} \frac{\partial^2 u}{\partial x^2}$ (rovnice vedení tepla).

6\. Užitím Γ - funkce vypočtěte

a) $\int_0^\infty x^{2n} e^{-x^2}\, dx$ $(n \in N)$ / $\frac{(2n-1)!!}{2^{n+1}} \sqrt{\pi}$ /

b) $\int_0^\infty e^{-x^n}\, dx$ $(n > 0)$ / $\frac{1}{n}\, \Gamma\left(\frac{1}{n}\right)$ /

c) $\int_0^\infty x^m e^{-x^n}\, dx$ $(m, n > 0)$ / $\frac{1}{n}\, \Gamma\left(\frac{m+1}{n}\right)$ /

d) $\int_0^1 \left(\ln \frac{1}{x}\right)^p dx$ / $\Gamma(p + 1)$ /

7. Najděte $\mathcal{L}\{f(t)\}$, je-li f(t)

a) $\sin^4 t$ / $\frac{1}{8}\left(\frac{s}{s^2+16} - \frac{4s}{s^2+4} + \frac{3}{s}\right.$ /

b) $\cos^6 t$ / $\frac{1}{32}\left(\frac{s}{s^2+36} + \frac{6s}{s^2+16} + \frac{15s}{s^2+4} + \frac{10}{s} \right)$ /

c) $\frac{1}{2}$ (ch at sin at + sh at cos at) / $\frac{as^2}{s^4+4a^4}$ /

d) $e^{-4t} \sin 3t \cos 2t$ / $\frac{1}{2}\left(\frac{5}{(s+4)^2+25} + \frac{1}{(s+4)^2+1}\right.$ /

e) t sh at sin at / $\frac{2a^2(3s^4-4a^4)}{(s^4+4a^4)^2}$ /

f) $f(t) = \begin{cases} 0 & (0 < t < a) \\ 1-a & (a < t < b) \\ b-a & (b < t) \end{cases}$ / $\left(\frac{1}{s^2} + \frac{a}{s} \right)\left(e^{-as} - e^{-bs} \right)$ /

8. Najděte $\Lambda\{F(s)\}$, je-li F(s)

a) $\frac{1}{s^2(s+1)^3}$ / $t - 3 + 3e^{-t} + 2t\,e^{-t} + \frac{1}{2}t^2 e^{-t}$ /

b) $\frac{s}{(s^2+1)(s^2+4)}$ / $\frac{1}{3}(\cos t - \cos 2t)$ /

c) $\frac{s+1}{s^2(s-1)(s+2)}$ / $-\frac{3}{4} - \frac{1}{2}t + \frac{2}{3}e^t + \frac{1}{12}e^{-2t}$ /

d) $\frac{1}{(s-1)^2(s-2)^3}$ / $\frac{1}{2}(t^2 e^{2t} - 4t\,e^{2t} + 6\,e^{2t} - 2t\,e^t - 6\,e^t)$ /

e) $\frac{3s}{(s^2+1)^2}$ / $\frac{3}{2}t \sin t$ / f) $\frac{s^2+a^2}{(s^2-a^2)^2}$ / t ch at /

g) $\frac{1}{(s^2+6s+13)(s^2+6s+10)}$ / $\frac{1}{6}e^{-3t}(2\sin t - \sin 2t)$ /

h) $\frac{s^2}{(s^2+4)(s^2+9)}$ / $\frac{1}{5}(3\sin 3t - 2\sin 2t)$ /

9. Řešte rovnice

a) $y' + ay = b$; $y(0) = 0$ / $\frac{b}{a}(1 - e^{-at})$ /

b) $y'' - y' - 6y = 2$; $y(0) = 1$, $y'(0) = 0$ / $\frac{1}{15}(12\,e^{-2t} + 8\,e^{3t} - 5)$ /

c) $y'' - 6y' + 9y = 0$; $y(0) = A$, $y'(0) = B$ / $A\,e^{3t} + (B - 3A)e^{3t}\,t$ /

d) $y'' - 4y = 4t$; $y(0) = 1$, $y'(0) = 0$ / $\frac{1}{4}(3\,e^{2t} + e^{-2t} - 4t)$ /

e) $y'' + 4y = 2\cos 2t$; $y(0) = 0$, $y'(0) = 4$ / $2\sin 2t + \frac{1}{2}t \sin 2t$ /

f) $y'' + y = \cos t + \sin 2t$; $y(0) = 0$, $y'(0) = 0$

/ $\frac{1}{6}(3t \sin t + 4 \sin t - 2 \sin 2t)$ /

g) $y''' + y' = 10\, e^{2t}$; $y(0) = y'(0) = y''(0) = 0$ / $e^{2t} + 4\cos t - 2\sin t - 5$ /

h) $y'' + 6y' + 13y = f(t)$; $y(0) = y'(0) = 0$ / $\frac{1}{2} \int_0^t f(t-\tau) e^{-3\tau} \sin 2\tau \, d\tau$ /

10. Řešte soustavy rovnic

a) $y' = 3z - y$ $\quad y(0) = z(0) = 0$ / $y(t) = \frac{3}{4} e^{2t} + \frac{1}{4} e^{-2t} - e^{t}$;

$z' = y + z + e^{t}$ $\quad z(t) = \frac{3}{4} e^{2t} - \frac{1}{12} e^{-2t} - \frac{2}{3} e^{t}$ /

b) $y' - 2y - 4z = \cos t$ $\quad y(0) = z(0) = 0$ / $y(t) = 4t + 2 - 2\cos t -$

$z' + y + 2z = \sin t$ $\quad - 3\sin t$; $z(t) = 2\sin t - 2t$ /

c) $y' + 7y - z = 0$ $\quad y(0) = z(0) = 1$ / $y = e^{-6t} \cos t$;

$z' + 2y + 5z = 0$ $\quad z = e^{-6t} (\cos t - \sin t)$ /

d) $y' - y + z = \frac{3}{2} t^{2}$ $\quad y(0) = z(0) = 0$ / $y = -\frac{1}{2} t^{2}$; $z = t^{2} + t$ /

$z' + 4y + 2z = 4t + 1$

e) $y' = -y + z + x$ $\quad y(0) = 1$, $z(0) = x(0) = 0$

$z' = y - z + x$ / $y = \frac{1}{3} e^{-t} + \frac{1}{2} e^{-2t} + \frac{1}{6} e^{2t}$; $z = \frac{1}{3} e^{-t} - \frac{1}{2} e^{-2t}$.

$x' = y + z + x$ $\quad + \frac{1}{6} e^{2t}$; $x = \frac{1}{3} e^{2t} - \frac{1}{3} e^{-t}$ /

f) $y'' + 2z = 0$ $\quad y(0) = 0$, $y'(0) = 1$, $z(0) = z'(0) = 0$

$z'' - 2y = 0$ / $y = \frac{1}{2} (\cos t \,\mathrm{sh}\, t + \sin t \,\mathrm{ch}\, t)$

$z = \frac{1}{2} (\sin t \,\mathrm{ch}\, t - \cos t \,\mathrm{sh}\, t)$ /