

Věta (Bolzanovo-Cauchyovo kritérium konvergence):

Reálná posloupnost $\{a_n\}$ je cauchyovská v $\mathbb{R} \Leftrightarrow$ je konvergentní v \mathbb{R} .

Důkaz: (přímý, stručná verze)

1) $\boxed{\Leftarrow} \quad \exists \lim_{n \rightarrow +\infty} a_n =: a \Rightarrow \forall \varepsilon > 0: |a_n - a| < \varepsilon \text{ pro s.v. } n$

$\boxed{? \quad \forall \bar{\varepsilon} > 0: |a_n - a_m| < \bar{\varepsilon} \text{ pro s.v. } n \text{ a s.v. } m ?}$

$$\begin{aligned} \forall \bar{\varepsilon} > 0: |a_n - a_m| &= |a_n - a + a - a_m| \leq \underbrace{|a_n - a|}_{< \frac{\bar{\varepsilon}}{2} \text{ pro s.v. } n} + \underbrace{|a - a_m|}_{< \frac{\bar{\varepsilon}}{2} \text{ pro s.v. } m} < \bar{\varepsilon} \text{ pro s.v. } n \text{ a s.v. } m \end{aligned}$$

2) $\boxed{\Rightarrow}$

$$\{a_n\} \text{ je cauchyovská} \Rightarrow \{a_n\} \text{ je omezená} \stackrel{(\text{B.-W.})}{\Rightarrow} \exists \{a_{n_k}\} : \exists \lim_{k \rightarrow +\infty} a_{n_k} =: a$$



$$\forall \varepsilon_1 > 0: |a_n - a_m| < \varepsilon_1 \text{ pro s.v. } n \text{ a s.v. } m$$



$$\forall \varepsilon_2 > 0: |a_{n_k} - a| < \varepsilon_2 \text{ pro s.v. } k$$

$\boxed{? \quad \forall \bar{\varepsilon} > 0: |a_n - a| < \bar{\varepsilon} \text{ pro s.v. } n ?}$

$$\begin{aligned} \forall \bar{\varepsilon} > 0: |a_n - a| &= |a_n - a_{n_k} + a_{n_k} - a| \leq \underbrace{|a_n - a_{n_k}|}_{< \frac{\bar{\varepsilon}}{2} \text{ pro s.v. } n \text{ a s.v. } k} + \underbrace{|a_{n_k} - a|}_{< \frac{\bar{\varepsilon}}{2} \text{ pro s.v. } k} < \bar{\varepsilon} \text{ pro s.v. } n \text{ a s.v. } k \end{aligned}$$

Důkaz: (přímý, podrobná verze)

$$1) \boxed{\Leftarrow} \quad \exists \lim_{n \rightarrow +\infty} a_n =: a \Rightarrow \forall \varepsilon > 0 \exists m_0 \in \mathbb{N} \forall n \in \mathbb{N}: n > m_0 \Rightarrow |a_n - a| < \varepsilon.$$

$$\text{Tedy } \forall \bar{\varepsilon} > 0 \exists \bar{m} \in \mathbb{N} \forall n \in \mathbb{N}: n > \bar{m} \Rightarrow |a_n - a| < \frac{\bar{\varepsilon}}{2},$$

$$\text{navíc } \forall m \in \mathbb{N}: m > \bar{m} \Rightarrow |a_m - a| < \frac{\bar{\varepsilon}}{2} \text{ (záměna } m \text{ za } n\text{).}$$

Protože

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m|,$$

dostáváme celkem

$$\forall \bar{\varepsilon} > 0 \exists \bar{m} \in \mathbb{N} \forall n \in \mathbb{N} \forall m \in \mathbb{N}: (n > \bar{m}) \wedge (m > \bar{m}) \Rightarrow |a_n - a_m| < \bar{\varepsilon}.$$

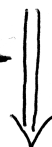
$\Rightarrow \{a_n\}$ je Cauchyovská v \mathbb{R} .

2) $\boxed{\Rightarrow}$

$$\{a_n\} \text{ je Cauchyovská v } \mathbb{R} \Rightarrow \{a_n\} \text{ je omezená} \xrightarrow{(\text{B.-W.})} \exists \{a_{m_k}\}: \exists \lim_{k \rightarrow +\infty} a_{m_k} =: a.$$



$\varepsilon > 0$ lib. pevně



$$\exists m_0 \in \mathbb{N}: |a_\mu - a_{m_k}| < \frac{\varepsilon}{2} \quad \forall \mu, k \in \mathbb{N}: \mu > m_0 \wedge m_k > m_0$$

$$\exists k_0 \in \mathbb{N}: |a_{m_k} - a| < \frac{\varepsilon}{2} \quad \forall k \in \mathbb{N}: k > k_0$$



$$|a_\mu - a| = |a_\mu - a_{m_k} + a_{m_k} - a| \leq |a_\mu - a_{m_k}| + |a_{m_k} - a| < \varepsilon \quad \forall \mu, k \in \mathbb{N}: (\mu > m_0) \wedge (k > k_0) \wedge (m_k > m_0)$$



$$\forall \mu \in \mathbb{N}: \mu > m_0 \Rightarrow |a_\mu - a| < \varepsilon$$



$$\lim_{\mu \rightarrow +\infty} a_\mu = a$$